ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 9, No. 4, 2014, *pp*. 267-276

A smoothing Newton method for the minimum norm solution of linear program

Lina Zhang , Zhensheng Yu, Yanyan Zhu

[1] *University of Shanghai for Science and Technology, Mobile: +86 18817849166;*
E-mail address: zhanglina19880806@126.com.

(*Received March 4, 2014, accepted October 1, 2014*)

Abstract. In this paper, we propose a smoothing Newton method to find the minimum norm solution of linear program problems. By using the smoothing technique, we reformulate the problem as an unconstrained minimization problem with a twice continuous differentiable objective function. The minimization of this objective function can be carried out by the classical Newton-type method which is shown to be globally convergence.

Keywords: Linear program, Minimum norm solution, Unconstrained minimization reformulation, Smoothing function, Newton-type method.

1. Introduction

Consider the linear program in primal form

$$\min c^T x \quad s.t. \quad Ax = b, x \geq 0 \tag{1}$$

Together with its dual

$$\max b^T \lambda \quad s.t. \quad A^T \lambda \leq c \tag{2}$$

Where $A \in R^{m \times n}, c \in R^n, and\ b \in R^m$ are the given data, and A is assumed to have full row rank. Let us denote the optimal value of the primal problem (1) by

$$\inf(\mathrm{P}) := \{\mathrm{c}^T \mathrm{x} \,|\, Ax = b, x \geq 0\}$$

Throughout this manuscript, we assume

$$\inf(P) \in R \tag{3}$$

This is equivalent to saying that the primal (and hence also the dual) linear program has a nonempty solution set.

The aim of this paper is to find the minimum norm solution of the primal program (1), i.e., we want to find the solution x^* of the program

$$\min \frac{1}{2}\|x\|^2 \quad s.t. \quad Ax = b, c^T x = \inf(\mathrm{P}), \mathrm{x} \geq 0 \tag{4}$$

Note that this problem has a unique solution under the assumption (3).Since the minimum norm solution could be a vertex as well as a point belonging to the relative interior of the solution set, neither the simplex method [1] nor the class of interior-point methods [2] will be assured to find the minimum norm solution of (1).

The standard method for finding a minimum norm solution of a convex program is based on the Tikhonov regularization[3]. Specialized to our linear program (1), the Tikhonov regularization generates a sequence of iterates $\{x^k\}$ with x^k being the unique solution of the regularized program

$$\min c^T x + \frac{\varepsilon_k}{2}\|x\|^2 \quad s.t. \quad Ax = b, x \geq 0 \tag{5}$$

where $\varepsilon_k > 0$ is a positive parameter and the sequence $\{\varepsilon_k\}$ tends to zero. However, the Tikhonov regularization is, in general, quite costly since it has to solve a sequence of quadratic programs. On the other hand, due to special properties of linear programs, it is known that a solution of a single quadratic program (5) with a sufficiently small but positive parameter already gives a solution of (4). This follows from Mangasarian and Meyer[[4],Corollary2].

To overcome this drawback, Kanzow, Qi and Qi [5] describes a new technique to solve problem (4), their main idea is to reformulate the problem as an unconstrained minimization problem with a convex and smooth objective function, a Newton-type method with global convergence was proposed. We note that although the objective function in [5] is smooth and convex, it does not twice continuous differentiable, and therefore the classical Newton method can't be used. Based on the recently development of smoothing function for complementarity problems, see, for examples [8,15,16,19], in this paper, we propose a smoothing unconstrained optimization reformulation and a Newton method to find the solution of problem (2). This reformulation provides a twice continuous differentiable objective function of the reformulation in [5], we obtain the positive definite of the Hessian matrix of the objective function under certain conditions and hence the classical Newton method can be used to solve the problem directly.

The organization of this paper is as follows. In Section 2, we give the smoothing unconstrained optimization reformulation and some properties of the reformulation including the Newton method. The numerical results are reported in Section 3.

A few words about our notation: We denote the n-dimensional real space by R^n . For a vector $x \in R^n$, we write $\|x\|$ for its Euclidean norm, and x_+ or $[x]_+$ for the vector $\max\{0, x\}$, where the maximum is taken componentwise, i.e., x_+ is the projection of x onto the nonnegative orthant. The rows of a matrix B will be denoted by B_i , where as we write b_{ij} for the (i, j) th element of the matrix B.

2. Unconstrained Minimization Reformulation

The following result is given in [6], which is essentially based on some related results given by Mangasarian [7] and Mangasarian and Meyer[4].

Theorem 2.1 A vector $x^* \in R^n$ is the minimum norm solution of the primal linear program (1) if and only if there exists a positive number R>0 such that, for each $r \geq R$, we have

$$x^* = \left[A^T \lambda_r^* - \mathrm{rc}\right]_+$$

Where λ_r^* denotes a solution of the nonlinear system

$$A\left[A^T \lambda_r - rc\right]_+ = b$$

Motivated by the characterization stated in Theorem 2.1, Kanzow, Qi and Qi [5] gave an unconstrained minimization reformulation of problem (4) as follows:

$$\min\ f(\lambda) = \frac{1}{2}\left\|\left[A^T \lambda_r - \mathrm{rc}\right]_+\right\|^2 - b^T \lambda_r,\ \lambda_r \in R^m \tag{6}$$

The objective function in (6) has the following property, the proof can be found in [5].

Lemma 2.1 The function f from (6) is convex and continuously differentiable with gradient:

$$\nabla f(\lambda_r) = A\left[A^T \lambda_r - rc\right]_+ - b \tag{7}$$

The lemma shows that x^* is the minimum norm solution of (1) if and only if λ_r^* is the stationary point of problem (6). The objective function f is once but not twice continuously differentiable, [5] employed a generalized Newton method to find the solution of problem (6).

In what follows, we consider reformulating the problem into a smoothing unconstrained optimization and then employ the classical Newton to solve it. Firstly, we introduce the smoothing function used in this paper. Define the step function

$$\sigma(x) = \begin{cases} 1 & if\ \ x > 0 \\ 0 & if\ \ x \leq 0 \end{cases}$$

In the extensive neural network literature, the step function is very effectively approximated by the sigmoid function

$$s(x,\alpha) = \frac{1}{1+e^{-\alpha x}}, \alpha > 0$$