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Finite Element Method for solving Linear Volterra Integro-Differential Equations of the second kind

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Abstract. In this paper, we present a method for numerical solution of linear Volterra integro - differential equations with boundary conditions. First, we obtain variational form of the problem, and then, finite element method and basis functions will be used. Also, the error analysis of the method is considered. Furthermore, the efficiency of the proposed method will be considered through numerical examples.

Keywords: Linear Volterra Integro-Differential Equations, Finite element method, Error estimation.

1. Introduction

Many authors have studied finite element methods for integral equations. See, Atkinson[2] , Ikebe [7], Nedelec [12], Sloan [16], and Wendland [19]. Adaptive finite element methods for integral equations have been considered more recently. See,[13,19].

Integro-differential equations have been discussed in many applied fields, such as biological, physical and engineering problems. They are usually difficult to solve analytically, so it is required to obtain an efficient approximate solution. There are several methods for solving integro-differential equations, Yanik and Fairweather in [20], used finite element methods for solving integro-differential equation of parabolic type and obtained an $O(h^{r+1}+(\Delta t)^2)$ order estimate for L^2 norm of the error.

In [9], Leroux and Thomee analyzed a Galerkin approximation in space with the Euler method in time for a semilinear integro-differential equations of parabolic type with non smooth data. The stability of Ritz-Volterra projections and error estimates for finite element methods for a class of integro-differential equations of parabolic type is studied by Lin and Zhang [10]. Sloan and Thomée, used time discretization of an integro-differential equation of parabolic type [17]. Brunner applied a collocation-type method to Volterra-Hammerstein integral equation as well as integro-differential equations, [3]. Volk used projection method to solve linear integro-differential equations, [18]. High order nonlinear Volterra Fredholm integro-differential equations has been solved in [11] by using Taylor polynomial. Sabri-Nadjafi [15] proposed He's variational iteration method for two systems of Volterra integro-differential equations.

In this paper, we use Lagrange polynomials with Finite element method to obtain an approximate solution of the problem. To illustrate the basic approach, we consider the following volterra integro-differential equation

$$-u''+b(x)u'(x)+c(x)u(x)=f(x)+\int_a^x K(x,t)u(t)dt u(a)=0, \quad u(b)=0,\ \Omega=[a,b] \tag{1}$$

We assume that $K(x,t)$ and $f(x)$ are continuous functions respect to their arguments, and $b(x)$ and $c(x)$ are nonnegative functions and belong to $C^1(\Omega)$. First, for using finite element method, by suitable linear transform, we convert the essential boundary condition to homogeneous one, and then we define

$$V=H_0^1(\Omega)=\{v\in H^1(\Omega), \quad v(a)=v(b)=0\}$$

where V is a Sobolev space together with following norm:

$$\|u\|_V^2=\|u\|_{L^2(\Omega)}^2+\|u'\|_{L^2(\Omega)}^2 .$$

For obtaining varational form, we let $B:V\times V\to R$ and $L:V\to R$ be bilinear form and linear functional, respectively.

The varational form of the problem is given as follows

$$B(u,v) = L(v), \quad \forall v \in V, \tag{2}$$

where

$$B(u,v) = \int_\Omega u'(x)v'(x)dx + \int_\Omega b(x)u'(x)v(x)dx + \int_\Omega c(x)u(x)v(x)dx - \int_\Omega v(x)(\int_a^b K(x,t)u(t)dt)dx L(v) = \int_\Omega f(x)v(x)dx \tag{3}$$

where $v(x) \in V$ is an arbitary function.

Lemma 1.1 Let B be bilinear form defined by (3). If $M_1 \le c(x) \le M_2$ and $P_1 \le b(x) \le P_2$, then B is continuous.

Proof. For B, we can write,

$$|B(u,v)| = |\int_\Omega u'(x)v'(x)dx + \int_\Omega b(x)u'(x)v(x)dx + \int_\Omega c(x)u(x)v(x)dx - \int_\Omega v(x)(\int_a^x K(x,t)u(t)dt)dx|$$

Using the Cauchy-Schwarz inequality and Sobolev norm, we have

$$|B(u,v)| \le \|u\|_{H^1}\|v\|_{H^1} + P_2\|u\|_{H^1}\|v\|_{H^1} + M_2\|u\|_{H^1}\|v\|_{H^1} + KR\|u\|_{H^1}\|v\|_{H^1} = (1+P_2+M_2+KR)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} = C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$$

which $K = \max_{\substack{a\le x\le b \\ a\le t\le x}} |K(x,t)|$, $R = \|1\|^2_{L^2(\Omega)}$ and $C = 1+P_2+M_2+KR$. So B is continous.

In addition of the hypothesis of lemma 1.1, suppose $0 \le b'(x) \le T_2$. Now we consider the V-ellipticity of B. For this purpose we write

$$\int_\Omega v'(x)v'(x)dx + \int_\Omega c(x)v(x)v(x)dx \ge \int_\Omega v'^2(x)dx \ge \frac{1}{1+c}\|v\|^2_{H^1}, \tag{4}$$

and

$$\int_\Omega b(x)v'(x)v(x)dx = \frac{-1}{2}\int_a^b b'(x)(v(x))^2dx \ge \frac{-T_2}{2}\int_a^b (v(x))^2dx \ge \frac{-T_2}{2}\|v\|^2_{H^1}, \tag{5}$$

also

$$-\int_\Omega v(x)(\int_a^x K(x,t)v(t)dt)dx \ge -|\int_\Omega v(x)(\int_a^x K(x,t)v(t)dt)dx| \ge -KR\|v\|^2_{L^2} \ge -KR\|v\|^2_{H^1}. \tag{6}$$

By ((4)), ((5)), ((6)), we have

$$B(v,v) \ge (\frac{1}{1+c} - \frac{T_2}{2} - KR)\|v\|^2_{H^1}, \tag{7}$$

or

$$B(v,v) \ge \alpha\|v\|^2_{H^1}, \tag{8}$$

where $\alpha = (\frac{1}{1+c} - \frac{T_2}{2} - KR)$, c is poincare's constant. So, the following lemma can be expressed.

Lemma 1.2 If $\alpha > 0$, B is V-elliptic.

By using Lax-Milgram theorem and lemmas 1.1, 1.2, the problem ((1)) has a unique solution.

2. Finite element method