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The use of radial basis functions for the solution of a partial differential equation with an unknown time-dependent coefficient

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Abstract. In this paper, a numerical technique is presented for the solution of a parabolic partial differential equation with a time-dependent coefficient subject to an extra measurement. For solving the discussed inverse problem, at first we transform it into a nonlinear direct problem and then use the proposed method. This method is a combination of collocation method and radial basis functions. The radial basis functions (RBFs) method is an efficient meshfree technique for the numerical solution of partial differential equations. The main advantage of numerical methods which use radial basis functions over traditional techniques is the meshless property of these methods. The accuracy of the method is tested in terms of maximum and RMS errors. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords: Radial basis functions, Inverse parabolic problems, Scattered data, Interpolation problem.

1. Introduction

Inverse problems are the problems that consist of finding an unknown property of an object, or a medium, from the observation of a response of this object, or medium, to a probing signal. Thus, the theory of inverse problems yields a theoretical basis for remote sensing and non-destructive evaluation. For example, if an acoustic plane wave is scattered by an obstacle, and one observes the scattered field far from the obstacle, or in some exterior region, then the inverse problem is to find the shape and material properties of the obstacle. Such problems are important in identification of flying objects (airplanes missiles, etc.), objects immersed in water (submarines, paces of fish, etc.) and in many other situations. In geophysics one sends an acoustic wave from the surface of the earth and collects the scattered field on the surface for various positions of the source of the field for a fixed frequency, or for several frequencies. The inverse problem is to find the subsurface inhomogeneities. In technology one measures the eigenfrequencies of a piece of a material, and the inverse problem is to find a defect in this material, for example, a hole in a metal. In geophysics the inhomogeneity can be an oil deposit, a cave, a mine. In medicine it may be a tumor or some abnormality in a human body. If one is able to find inhomogeneities in a medium by processing the scattered field on the surface, then one does not have to drill a hole in a medium. This, in turn, avoids expensive and destructive evaluation. The practical advantages of remote sensing are what make the inverse problems important [6].

The parameter identification in a parabolic differential equation from the overspecified data plays an important role in engineering and physics [1-3]. This technique has been widely used to determine the unknown properties of a region by measuring only data on its boundary or a specified location in the domain. These unknown properties such as the conductivity medium are important to the physical process but usually cannot be measured directly or very expensive to be measured [4, 5].

In this paper we shall consider an inverse problem of finding an unknown parameter $q(t)$ in a parabolic partial differential equation. The classical example is that one needs to find the temperature distribution $u(x,t)$ as well as the thermal coefficient $q(t)$ simultaneously that satisfy:

$$u_t = u_{xx} + q(t)u_x + f(x,t); \qquad 0 < x < 1, \quad 0 < t < T, \tag{1}$$

with the initial-boundary conditions:

$$u(x,0) = u_0(x); \qquad 0 \le x \le 1, \tag{2}$$

$$u(0,t) = g_0(t); \qquad 0 \le t \le T, \tag{3}$$

$$u(1,t) = g_1(t); \qquad 0 \le t \le T, \tag{4}$$

and subject to an extra measurement:

$$\int_0^1 u(x,t)dx = E(t); \qquad 0 \le t \le T, \tag{5}$$

where $T > 0$ is constant and f, u_0, g_0, g_1 and E are known functions.

The existence and uniqueness of this inverse problem is discussed in [7, 8] and to interpret the integral equation (5), the reader can refer to [9, 10].

There is a fundamental difference between the direct and the inverse problems. In all cases, the inverse problem is ill-posed or improperly posed in the sense of Hadamard, while the direct problem is well-posed. A mathematical model for a physical problem is called as well-posed in the sense that it has the following three properties:

- There exists a solution of the problem (existence).
- There is at most one solution of the problem (uniqueness).
- The solution depends continuously on the data (stability).

Thus an important task is to formulate the problem properly and to find the conditions that ensure its well posedness. If the solution of the given problem exists and is unique but it does not depend continuously on the data, then in general the computed solution has nothing to do with the true solution. The ill-posedness may be a main difficulty for the inverse problems. Since it is hard to avoid some errors in the observation $E(t)$ which is obtained from experiments, a small perturbation in $E(t)$ may result in a big change in $q(t)$ which may make the obtained results meaningless [11, 12].

In this paper, we solve this problem by using radial basis functions (RBFs) as a truly meshless method. In a meshless (meshfree) method a set of scattered nodes is used instead of meshing the domain of the problem. The use of radial basis functions as a meshless method for numerical solution of partial differential equations is based on the collocation method. Because of the collocation technique, this method does not need to evaluate any integral. The main advantage of numerical methods, which use radial basis functions over traditional techniques, is the meshless property of these methods. The radial basis functions method is used actively for solving partial differential equations. For example see [14-19]. Also some applications of this approach in solving inverse problems can be found in [20-23].

The organization of this article is as follows. In Section 2, we describe radial basis functions and its properties. In Section 3, the presented technique is used to approximate the solution of the inverse problem (1)-(5). In Section 4, an error analysis is presented. In Section 5, we give some computational results of numerical experiments with RBFs method to support our theoretical discussion. The conclusions are discussed in Section 6.

2. Radial basis function approximation

The problem of interpolating functions of d real variables ($d > 1$) occurs naturally in many areas of applied mathematics and the sciences. Radial basis functions method can provide interpolants to function values given at irregularly positioned points for any value of d. Further, these interpolants are often excellent approximations to the underlying function, even when the number of interpolation points is small. Although polynomials (e.g., Chebyshev and Legendre) are very powerful tools for interpolating a set of points in one-dimensional domains, the use of these functions is not efficient in higher-dimensional or irregular domains. In the use of these functions, the points in the domain of the problem should be chosen in a special form, which is very limiting when the interpolation of a scattered set of points is needed. The main advantage of RBFs is that it requires neither domain nor surface discretization, so the method is independent of the dimension of the problem. The method is meshless and is not complicated. Some meshless schemes are the diffuse element method [24], the partition of unity method [25], the element-free Galerkin method [26], the reproducing kernel particle method [27], the finite point method [28], the meshless local Petrov-Galerkin method [29] and the general finite difference method [30].

Let $\mathbb{R}^+ = \{x \in \mathbb{R}, x \ge 0\}$, $\|\cdot\|_2$ denotes the Euclidean norm and $\varphi : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function