ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 9, No. 4, 2014, *pp*. 310-320

Identification of the unknown diffusion coefficient in a parabolic equation using HPM

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(*Received September 19, 2014, accepted October 20, 2014*)

Abstract. In this paper, the homotopy perturbation method (HPM) is proposed to solve an inverse problem of finding an unknown function in parabolic equation with an extra measurement. For solving the discussed inverse problem, at first we transform it's into a nonlinear direct problem then uses the proposed method. Also an error analysis is presented for the method and prior and posterior error bounds of the approximate solution are estimated. Application of the HPM to this problem shows the rapid convergence of the sequence constructed by this method to the exact solution.

Keywords: Homotopy perturbation method (HPM), inverse parabolic problem, extra measurement.

1. Introduction

The homotopy perturbation method was first proposed by the Chinese mathematician Ji-Huan He [1-5]. The method has been used by many authors to handle a wide variety of scientific and engineering applications to solve various functional equations. In this method, the solution is considered as the sum of an infinite series, which converges rapidly to accurate solutions. One of the most remarkable features of the HPM is that usually just few perturbation terms are sufficient for obtaining a reasonably accurate solution. Considerable research works have been conducted recently in applying this method to a class of linear and nonlinear equations. This method was further developed and improved by He, and applied to nonlinear oscillators with discontinuities [5], nonlinear wave equations [6], boundary value problems [7], limit cycle and bifurcation of nonlinear problems [8], nonlinear Schrodinger equations [9], nonlinear equations arising in heat transfer [10] quadratic Ricatti differential equation [11] Klein-Gordon equation [12] and Blasius equation [13].

In this paper, we propose HPM to solve the inverse problem of finding the function $u(x,t)$ and the unknown positive diffusion coefficient $\mathrm{a}(t)$ in the parabolic initial-boundary value problem as following:

$$u_t = a(\mathrm{t})u_{xx} + p(t)u + g(x,t); \qquad 0 < x < 1, \quad 0 < t < T, \tag{1}$$

subject to the initial and boundary conditions:

$$u(x,0) = u_0(x); \qquad 0 \le x \le 1, \tag{2}$$

$$u(0,\mathrm{t}) = g_0(\mathrm{t}); \qquad 0 \le t \le T, \tag{3}$$

$$u(1,\mathrm{t}) = g_1(\mathrm{t}); \qquad 0 \le t \le T, \tag{4}$$

where $T > 0$ and g, p, u_0, g_0 and g_1 are known functions.

An additional boundary condition which can be the additional specification at a point in the spatial domain (temperature additional specification), is given in the following form:

$$u(\mathrm{x}^*,\mathrm{t}) = E(\mathrm{t}); \qquad 0 \le t \le T, \tag{5}$$

where E is known function and $x^* \in (0,1)$ is constant. Employing the condition (5), a recovery of the function $\mathrm{a}(t)$ together with the solution $u(x,t)$ can be made possible.

Therefore in this study, we solve the inverse problem (1)-(5).

Certain types of physical problems can be modeled by (1)-(5). As is said in [18], one application is in the determination of the unknown properties in a region by measuring only data on the boundary and particular attention has been focused to coefficients that present physical meaning quantities. For example, the conductivity of a medium.

The existence and uniqueness of the solution of this problem and more applications are discussed in

[14-17]. However, the theory of the numerical solution of this problem is far from satisfactory. In [18], a backward Euler finite difference scheme was discussed. Authors of [19] proved the determination of a time-dependent conductivity is possible for an arbitrary domain in $\mathbb{R}^n$ in a well-posed manner. In [20], this problem was studied from a different point of view. The authors first transformed a large class of parabolic inverse problems into a non-classical parabolic equation whose coefficients consist of trace type functional on the solution and its derivatives subject to some initial and boundary conditions. For the resulted non-classical problem, they introduced a variational form by defining a new function, then both continuous and discrete Galerkin procedures are employed to the non-classical problem. Author of [21] used the several explicit and implicit finite difference methods to solve this problem. In [22], an efficient pseudospectral Legendre method is developed to solve problem (1)-(5). Also, a method is proposed in [23] to solve this problem which is based on a semi-analytical approach. In [24], the numerical solution is also considered by use of Chebyshev cardinal functions. This problem is solved by a high-order compact finite difference method in [25].

2. Homotopy perturbation method

To illustrate the basic idea of this method [1-13], we consider the following nonlinear differential equation:

$$\mathrm{A(u)-f(r)=0}; \qquad r\in\Omega. \tag{6}$$

Considering the boundary condition of:

$$\mathrm{B}(\mathrm{u},\frac{\partial u}{\partial n})=0; \qquad r\in\Gamma, \tag{7}$$

where A is a general differential operator, B a boundary operator, $f(\mathrm{r})$ a known analytical function and Γ is the boundary of the domain $\Omega\subset\mathbb{R}^d$; $d=1,2,3$. Generally speaking, the operator A can be divided into two parts which are L and N, where L is a simple part which is easy to handle and N contains the remaining parts of A. Therefore equation (6) can be rewritten as follows:

$$\mathrm{L(u)+N(u)-f(r)=0}. \tag{8}$$

Using the homotopy technique, we construct a homotopy as $v(\mathrm{r,p}):\Omega\times[0,1]\to\mathbb{R}$ which satisfies:

$$H(\mathrm{v,p})=(1-\mathrm{p})[\mathrm{L(v)-L(u_0)}]+\mathrm{p}[\mathrm{A(v)-f(r)}]=0; \qquad p\in[0,1], \tag{9}$$

or

$$H(\mathrm{v,p})=\mathrm{L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]}=0; \qquad p\in[0,1], \tag{10}$$

where p is an embedding parameter and u_0 is an initial approximation of equation (6). Clearly, we have:

$$H(\mathrm{v},0)=\mathrm{L(v)-L(u_0)}=0, \tag{11}$$

$$H(\mathrm{v},1)=A(\mathrm{v})-f(\mathrm{r})=0. \tag{12}$$

The changing process of p from zero to unity is just that of $v(\mathrm{r,p})$ from $u_0(\mathrm{r})$ to $u(\mathrm{r})$. In topology, this is called deformation, and $\mathrm{L(v)-L(u_0)}$ and $A(\mathrm{v})-f(\mathrm{r})$ are called homotopy. According to HPM, we can first use the embedding parameter p as a "small parameter" and assume that the solution of equations (9) and (10) can be written as a power series in p:

$$v=u_0+pu_1+p^2u_2+\ldots. \tag{13}$$

Setting $p=1$ results the approximate solution of equation (6):

$$u=\lim_{p\to1}v=u_0+u_1+u_2+\ldots. \tag{14}$$

For nonlinear term N in equation (8), we can write:

$$N(\mathrm{v})=\mathrm{N(u_0)+pN(u_0,u_1)}+\cdots=\sum_{n=0}^{\infty}p^nN(\mathrm{u_0,u_1,\cdots,u}_n), \tag{15}$$

where $N(\mathrm{u_0,u_1,\cdots,u}_n)$ is called He's polynomial [26] defined by: