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Analytical Treatment of Abel Integral Equations by Optimal Homotopy Analysis Transform Method

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Abstract. In this paper, a user friendly algorithm based on the optimal homotopy analysis transform method (OHATM) is proposed to solve a system of generalized Abel's integral equations. The classical theory of elasticity of material is modeled by the system of Abel integral equations. It is observed that the approximate solutions converge to the exact solutions. Illustrative numerical examples are given to demonstrate the efficiency and simplicity of the proposed method in solving such types of systems of Abel's integral equations. Finally, several numerical examples are given to illustrate the accuracy and stability of this method. Comparison of the approximate solution with the exact solutions we show that the proposed method is very efficient and computationally attractive.

Keywords: integeral equation, Abel integral equation, optimal homotopy analysis transform method, Laplace transform.

1. Introduction

An integral equation is defined as equations in which the unknown function $y(x)$ to be determined appear under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations. Abel's equation is one of the integral equations derived directly from a concrete problem of physics, without passing through a differential equation. This integral equation occurs in the mathematical modeling of several models in physics, astrophysics, solid mechanics and applied sciences. The great mathematician Niels Abel, gave the initiative of integral equations in 1823 in his study of mathematical physics [1-4]. In 1924, generalized Abel's integral equation on a finite segment was studied by Zeilon [5]. The different types of Abel integral equation in physics have been solved by Pandey et al. [6], Kumar and Singh [7], Kumar et al. [8], Dixit et al. [9], Yousefi [10], Khan and Gondal [11], Li and Zhao [12] by applying various kinds of analytical and numerical methods.

The main aim of this article is to present analytical and approximate solution of integral equations by using new mathematical tool like optimal homotopy analysis transform method. The proposed method is coupling of the homotopy analysis method HAM and Laplace transform method. The HAM, first proposed in 1992 by Liao, has been successfully applied to solve many problems in physics and science [13-18]. In recent years many authors have paid attention to study the solutions of linear and nonlinear partial differential equations by using various methods combined with the Laplace transform [19-27]. A typical form of an integral equation in $y(x)$ is of the form:

$$\mathbf{y(x)} = \mathbf{f(x)} + \lambda \int_{\alpha(x)}^{\beta(x)} \mathbf{K(x,t)y(t)dt}, \tag{1}$$

where $K(x,t)$ is called the kernel of the integral equation (1), and $\alpha(x)$ and $\beta(x)$ are the limits of integration. It can be easily observed that the unknown function $y(x)$ appears under the integral sign. It is to be noted here that both the kernel $K(x,t)$ and the function $f(x)$ in equation (1) are given functions; and λ is a constant parameter. The prime objective of this text is to determine the unknown function $y(x)$ that will satisfy equation (1) using a number of solution techniques. We shall devote considerable efforts in exploring these methods to find solutions of the unknown function

2. Bounded extended Cesàro operators

In order to elucidate the solution procedure of the optimal homotopy analysis transform method, we consider the following integral equations of second kind:

$$\mathbf{y(x) = f(x) + \int_0^x K(x,t)y(t)dt, 0 \le x \le 1} \tag{2}$$

Now operating the Laplce transform on both sides in Eq. (2), we get

$$\mathbf{L[y(x)] = L[f(x)] + L\left\{\int_0^x K(x,t)y(t)dt\right\}} \tag{3}$$

We define the nonlinear operator

$$\mathbf{N[\phi(x;q)] = L[\phi(x;q)] - L[f(x)] - L\left\{\int_0^x K(x,t)\phi(x;q)dt\right\}} \tag{4}$$

where $q \in [0,1]$ be an embedding parameter and $\boldsymbol{\phi(x;q)}$ is the real function of x and q. By means of generalizing the traditional homotopy methods, the great mathematician Liao [13-14] construct the zero order deformation equation

$$(\mathbf{1-q})\mathbf{L}[\boldsymbol{\phi}(\mathbf{x;q}) - \mathbf{y}_0(x)] = \hbar\mathbf{q}\mathbf{H}(x)\mathbf{N}[\boldsymbol{\phi}(\mathbf{x;q})], \tag{5}$$

where is a nonzero auxiliary parameter, $\mathbf{H}(x) \neq 0$ an auxiliary function, $\mathbf{y_0(x)}$ is an initial guess of $\mathbf{y(x)}$ and $\boldsymbol{\phi(x;q)}$ is an unknown function. It is important that one has great freedom to choose auxiliary thing in OHATM. Obviously, when $q=0$ and $q=1$, it holds

$$\boldsymbol{\phi}(\mathbf{x;0}) = \mathbf{y}_0(x), \boldsymbol{\phi}(\mathbf{x;1}) = \mathbf{y}(x), \tag{6}$$

Respectively. Thus, as q increases from 0 to 1, the solution varies from the initial guess to the solution. Expanding $\phi(x;q)$ in Taylor's series with respect to q, we have

$$\boldsymbol{\phi}(\mathbf{x;q}) = \mathbf{y}_0(\mathbf{x,t}) + \sum_{\mathbf{m=1}}^{\infty} \mathbf{q}^m \mathbf{y}_m(\mathbf{x}), \tag{7}$$

where

$$\mathbf{y}_m(\mathbf{x}) = \frac{1}{m!}\frac{\partial^m \boldsymbol{\phi}(\mathbf{x;q})}{\partial q^m}\Big|_{q=0} \tag{8}$$

If the auxiliary linear operator, the initial guess , the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen , the series (7) converges at $q=1$, we have

$$\mathbf{y(x)} = \mathbf{y}_0(\mathbf{x}) + \sum_{\mathbf{m=1}}^{\infty} \mathbf{y}_m(\mathbf{x}), \tag{9}$$

which must be one of the solutions of the original integral equations. Define the vectors

$$\overleftrightarrow{y}_n = \{\mathbf{y}_0(x), \mathbf{y}_1(x), \ldots \mathbf{y}_n(x)\}. \tag{10}$$

Differentiating equation (6) m -times with respect to the embedding parameter q, then setting $q=0$ and finally dividing them by $m!$, we obtain the m^{th} -order deformation equation.