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# On the Characterization of Nonuniform Wavelet Sets on Positive Half Line

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**Abstract.** In this paper, we study the characterization of nonuniform wavelet sets on positive half line and we also prove the existence of nonuniform wavelet sets associated with the dilation *N* and the translation set $\Lambda^+_{r,N} = \left\{0, \frac{r}{N}\right\} + \mathbb{Z}^+, N > 1$ (an integer) and $r$ is an odd integer with $1 \le r \le 2N-1$ such that $r$ and $N$ are relatively prime and $\mathbb{Z}^+$ is the set of non-negative integers.

**Keywords:** Nonuniform Wavelets and nonuniform wavelet set

## 1. Introduction

The concepts of wavelet and multiresolution analysis has been extended to many different setups. One can replace the dilation factor 2 by an integer $M \ge 2$ and one needs to construct $N-1$ wavelets to generate the whole space $L^2(\mathbb{R})$. In general, in higher dimensions, it can be replaced by a dilation matrix $A$, in which case the number of wavelets required is $|\det A| - 1$. But in all these cases, the translation set is always a group. In the two papers [5, 6], Gabardo and Nashed considered a generalization of Mallat's [13] celebrated theory of MRA, in which the translation set acting on the scaling function associated with the MRA to generate the subspace $V_0$ is no longer a group, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. More precisely, this set is of the form $\left\{0, \frac{r}{N}\right\} + 2\mathbb{Z}$, where $N \ge 1$ is an integer, $1 \le r \le 2N-1$, $r$ is an odd integer relatively prime to $N$. They call this a nonuniform multiresolution analysis (NUMRA) and is based on the theory of spectral pairs. Farkov [3] has given general construction of compactly supported orthogonal *p*-wavelets in $L^2(\mathbb{R}^+)$. Farkov et al. [4] gave an algorithm for biorthogonal wavelets related to Walsh functions on positive half line. Shah [15], studied the construction of *p*-wavelet packets associated with the multiresolution analysis defined by Farkov [3], for $L^2(\mathbb{R}^+)$. Meenakshi et al. [14] studied NUMRA on positive half line. Recently, Shah and Abdullah [16], have constructed nonuniform multiresolution analysis on local fields of positive characteristic and proved the necessary and sufficient condition for the existence of associated wavelets.

In the present paper, we study characterization of nonuniform wavelet sets on positive half line and we also prove the existence of nonuniform wavelet sets associated with the dilation $N$ and the translation set $\Lambda^+_{r,N} = \left\{0, \frac{r}{N}\right\} + \mathbb{Z}^+, N > 1$ is an integer and $r$ is an odd integer with $1 \le r \le 2N-1$ such that $r$ and $N$ are relatively prime. This paper is organized as follows. In Sec. 2, we present a brief review of generalized Walsh functions and polynomials, the Walsh-Fourier transform. In Sec. 3, we study characterization for nonuniform wavelet sets and proves their existence.

## 2. Notations and preliminaries

Throughout we shall denote $|A|$ and $\chi_A(\xi)$, respectively Lebesgue measure and characteristic function of $A$. If $N \ge 1$ is an integer, we define

$$\Gamma^+_N = \{mN + j : m \in \mathbb{Z}^+, \quad j = 0, 1, \dots, N-1\}$$

and if $r$ is any odd integer with $1 \le r \le 2N-1$ such that $r$ and $N$ are relatively prime, the set $\Lambda^+_{r,N} = \left\{0, \frac{r}{N}\right\} + \mathbb{Z}^+$.

## 2.1. Walsh-Fourier Analysis

Let $p$ be a fixed natural number greater than 1. As usual, let $\mathbb{R}^+ = [0,\infty)$ and $\mathbb{Z}^+ = \{0,1,\dots\}$. Denote by $[x]$ the integer part of $x$. For $x \in \mathbb{R}^+$ and for any positive integer $j$, we set

$$x_j = [p^j x](\mathrm{mod}\, p), \qquad x_{-j} = [p^{1-j}x](\mathrm{mod}\, p), \tag{2.1}$$

where $x_j,\ x_{-j} \in \{0,1,\dots,p-1\}$.

Consider the addition defined on $\mathbb{R}^+$ as follows:

$$x \oplus y = \sum_{j<0} \xi_j p^{-j-1} + \sum_{j>0} \xi_j p^{-j} \tag{2.2}$$

with

$$\xi_j = x_j + y_j (\mathrm{mod}\, p), \qquad j \in \mathbb{Z} \setminus \{0\}, \tag{2.3}$$

where $\xi_j \in \{0,1,\dots,p-1\}$ and $x_j,\ y_j$ are calculated by (2.1). Moreover, we write $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo $p$ in $\mathbb{R}^+$.

For $x \in [0,1)$, let $r_0(x)$ be given by

$$r_0(x) = \begin{cases} 1, & x \in \left[0, \dfrac{1}{p}\right) \\ \epsilon_p^j, & x \in [jp^{-1}, (j+1)p^{-1}),\ j = 1,2,\dots,p-1, \end{cases} \tag{2.4}$$

where $\epsilon_p = \exp\left(\frac{2\pi i}{p}\right)$. The extension of the function $r_0$ to $\mathbb{R}^+$ is defined by the equality $r_0(x+1) = r_0(x),\ x \in \mathbb{R}^+$. Then the generalized Walsh functions $\{\omega_m(x)\}_{m \in \mathbb{Z}^+}$ are defined by

$$\omega_0(x) = 1, \qquad \omega_m(x) = \prod_{j=0}^{k} \left(r_0(p^j x)\right)^{\mu_j},$$

where $m = \sum_{j=0}^{k} \mu_j p^j,\ \mu_j \in \{0,1,2,\dots,p-1\}, \mu_k \neq 0$.

For $x,\ \omega \in \mathbb{R}^+$, let

$$\chi(x,\omega) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j \omega_{-j} + x_{-j}\omega_j)\right), \tag{2.5}$$

where $x_j$ and $\omega_j$ are calculated by (2.1).

We observe that

$$\chi\left(x, \frac{m}{p^{n-1}}\right) = \chi\left(\frac{x}{p^{n-1}}, m\right) = \omega_m\left(\frac{x}{p^{n-1}}\right) \qquad \forall x \in [0, p^{n-1}), \qquad m \in \mathbb{Z}^+.$$

The Walsh-Fourier transform of a function $f \in L^1(\mathbb{R}^+)$ is defined by