ACADEMIC
World Academic Union

ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 10, No. 2, 2015, *pp*. 090-097

A parallel descent-like prediction-correction method for structured variational inequalities with three blocks

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(*Received November 08, 2014, accepted March 07, 2015*)

Abstract. This paper proposes a parallel descent-like prediction-correction method(PDPCM) for structured variational inequalities with three blocks. In the prediction step, the intermediate point $(\bar{x},\bar{y},\bar{z},\bar{\lambda})$ is produced by solving three low-dimensional variational inequalities in parallel. In the correction step, a descent direction is first constructed using the iterate and the intermediate point, and then the new iterate is obtained along this descent direction. Global convergence of the proposed method is proved under mild assumptions, and we also establish its worse-case $O(1/t)$ convergence rate in the ergodic sense.

Keywords: variational inequality problems, prediction-correction method, global convergence, convergence rate.

1. Introduction

Let $S \subseteq R^n$ be a nonempty closed convex set and F be a continuous mapping from S into itself. The variational inequality problem, denoted by VI(S,F), is to find $u^* \in S$, such that

$$(u-u^*)^T F(u^*) \geq 0, \forall u \in S, \tag{1}$$

where \`T' denotes the standard inner product. In this paper, we consider the VI(S,F) which has the following separable structure:

$$u=\begin{pmatrix} x \\ y \\ z \end{pmatrix}, F(u)=\begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix}$$

$$S=\{(x,y,z) \mid x\in X, y\in Y, z\in Z, Ax+By+Cz=b\},$$

where $X \subseteq R^{n_1}, Y \subseteq R^{n_2}, Z \subseteq R^{n_3}$, , are given nonempty closed convex sets; $A\in R^{m\times n_1}, B\in R^{m\times n_2}$, $C\in R^{m\times n_3}$ are given matrices with full rank; $b\in R^n$ is a given vector, and $f: R^{n_1}\to R^{n_1}, g: R^{n_2}\to R^{n_2}, h: R^{n_3}\to R^{n_3}$, are given monotone continuous mappings.

By attaching a Lagrangian multiplier $\lambda \in R^m$ to the linear constraint $Ax+By+Cz=b$, the above variational inequality problem can be converted into the following form (denoted by VI(W,Q): Find $w^*=(x^*,y^{*},z^*,\lambda^*)\in W := X\times Y\times Z\times R^m$ such that

$$(w-w^*)^T Q(w^*) \geq 0, \forall x \in W, \tag{2}$$

Where

$$w=\begin{pmatrix} x \\ y \\ z \\ \lambda \end{pmatrix}, Q(w)=\begin{pmatrix} f(x)-A^T\lambda \\ g(y)-B^T\lambda \\ h(z)-C^T\lambda \\ Ax+By+Cz-b \end{pmatrix}$$

To solve VI (W,Q), the proximal alternating direction method(PADM)[2,5] finds a new iterate $(x^{k+1},y^{k+1},z^{k+1},\lambda^{k+1})\in W$ from a given iterate $(x^k,y^k,z^k,\lambda^k)\in W$ via the following procedure: Find $\tilde{x}^k\in X$ such that

$$(x'-\tilde{x}^k)^T\{f(\tilde{x}^k)-A^T[\lambda^k-\beta(A\tilde{x}^k+By^k+Cz^k-b)]+r(\tilde{x}^k-x^k)\}\ge 0,\forall x'\in X \quad (3)$$

and take $\tilde{x}^k$ as x^{k+1}. Find $\tilde{y}^k\in Y$ such that

$$(y'-\tilde{y}^k)^T\{g(\tilde{y}^k)-B^T[\lambda^k-\beta(A\tilde{x}^k+B\tilde{y}^k+Cz^k-b)]+s(\tilde{y}^k-y^k)\}\ge 0,\forall y'\in Y \quad (4)$$

and take $\tilde{y}^k$ as y^{k+1}. Find $\tilde{z}^k\in Z$ such that

$$(z'-\tilde{z}^k)^T\{h(\tilde{z}^k)-C^T[\lambda^k-\beta(A\tilde{x}^k+B\tilde{y}^k+C\tilde{z}^k-b)]+t(\tilde{z}^k-z^k)\}\ge 0,\forall z'\in Z \quad (5)$$

and take $\tilde{z}^k$ as z^{k+1}. Finally, update λ^k via

$$\lambda^{k+1}=\lambda^k-\beta(A\tilde{x}^k+B\tilde{y}^k+C\tilde{z}^k-b) \quad (6)$$

where $r\ge 0,s\ge 0,t\ge 0$ are given proximal parameters; $\beta>0$ is a given penalty parameter for the linear constraint $Ax+By+Cz=b$. Note that, when $r=s=t=0$ in (3)-(5), the classical alternating direction method is obtained. Obviously, the variables $(\tilde{x},\tilde{y},\tilde{z})$ of the involved low-dimensional variational inequalities (3)-(5) are crossed and thus the PADM is not eligible for parallel computing in the sense that the solutions of sub-variational inequalities (3)-(5) cannot be obtained simultaneously. For the purpose of parallel computing, the author of [3] proposed the following parallel form of the PADM, denoted by PPADM (in fact, in [3], $r=s=t=0$): Find $\tilde{x}^k\in X$ such that

$$(x'-\tilde{x}^k)^T\{f(\tilde{x}^k)-A^T[\lambda^k-\beta(A\tilde{x}^k+By^k+Cz^k-b)]+r(\tilde{x}^k-x^k)\}\ge 0,\forall x'\in X \quad (7)$$

Find $\tilde{y}^k\in Y$ such that

$$(y'-\tilde{y}^k)^T\{g(\tilde{y}^k)-B^T[\lambda^k-\beta(Ax^k+B\tilde{y}^k+Cz^k-b)]+s(\tilde{y}^k-y^k)\}\ge 0,\forall y'\in Y \quad (8)$$

Find $\tilde{z}^k\in Z$ such that

$$(z'-\tilde{z}^k)^T\{h(\tilde{z}^k)-C^T[\lambda^k-\beta(Ax^k+By^k+C\tilde{z}^k-b)]+t(\tilde{z}^k-z^k)\}\ge 0,\forall z'\in Z \quad (9)$$

Finally, update λ^k via

$$\lambda^{k+1}=\lambda^k-\beta(A\tilde{x}^k+B\tilde{y}^k+C\tilde{z}^k-b) \quad (10)$$

Obviously, the variables of the above sub-variational inequalities are not crossed. However, we cannot ensure convergence of the iterate sequence if we set $w^{k+1}=\tilde{w}^k$, therefore, in [3], the new iterate w^{k+1} is generated by a descent step. More specifically,

$$w^{k+1}=w^k-\alpha_k H^{-1}M(w^k-\tilde{w}^K), \quad (11)$$

where M is a predefined symmetric matrix; H is a given proper positive definite matrix, and α_k is a judiciously chosen step size.

The sub-variational inequalities (7)-(9) are easy to solve if the evaluations of $(A^TA+f/\beta)^{-1}(Av)$, $(B^TB+g/\beta)^{-1}(Bv)$, and $(C^TC+h/\beta)^{-1}(Cv)$, are simple, however, if A,B and C are not identity matrices, the above evaluations could be costly. In this paper, motivated by [6], we propose a modified version of the PPADM (denoted by PDPCM), and the new sub-variational inequalities in PDPCM are easy to solve under the assumption that the resolvent operators of f,g and h are easy to evaluate, where the resolvent operator of mapping T is defined as $(I+\lambda T)^{-1}(v)$. We prove the global convergence of the PDPCM, moreover, inspired by the strategy in [4,5], we establish its worst-case $O(1/t)$ convergence rate in ergodic sense.

The rest of the paper is organized as follows. We first give some basic concepts which are useful in the following analysis in Section 2. Then, in Section 3, we describe the parallel descent-like prediction-correction method(PDPCM) for structured variational inequalities, and the global convergence of the new method is proved. We establish the PDPCM's worst-case $O(1/t)$ convergence rate in ergodic sense in Section 4 and some conclusions are drawn in the last section.