

Numerical Solution of Fractional Differential Equations by using Fractional Spline Model

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Abstract. In this paper, we consider a new suitable lacunary fractional interpolation with the idea of the spline function of polynomial form, and the method applied to solve linear fractional differential equations. The results obtained are in good agreement with the exact analytical solutions and the numerical results presented by several examples, results also show that the technique introduced here is robust and easy to apply.

Keywords: Fractional integral and derivative; Caputo Derivative; Taylor's expansion; Error bound; Spline functions

1. Introduction

Fractional differential equations are gaining considerable importance recently due to their wide range of applications in the fields of Physics, Engineering [3, 19], Chemistry, and/or Biochemistry [4], Control [5,6,7, 8], Medicine [9] and Biology [1]. Several techniques such as Adomian decomposition method (ADM) [11], Adams-Bashforth- Moulton method [14, 15], Fractional difference method [10], and Variational iteration method [12, 13] have been developed for solving non linear functional equations in general and solving fractional differential equations in particular.

In view of successful application of spline functions of polynomial form in system analysis [22], fractional differential equations ([24], [25]), and delay differential equations of fractional order [23], we hold that it should be applicable to solve fractional differential equations with the idea of the lacunary interpolation. For more details on lacunary interpolation we may refer to ([16]–[18]).

In this paper we shall apply fractional spline to find the approximate analytical solution of the fractional differential equation. Error bound and existence and uniqueness for the method will be performed.

2. Preliminaries

In this section, some definitions and Taylor's Theorem, used in our work, will be presented. There are many definitions for fractional derivatives, the most commonly used ones are the Riemann-Liouville and the Caputo derivatives, especially the Caputo derivative are involved in our work. Suppose that $\alpha > 0, x > a, a, x \in \mathbb{R}$, then

Definition 1[21] The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \xi)^{\alpha-1} f(\xi) d\xi, \quad n-1 < \alpha < n \in \mathbb{N},$$

where Γ is the gamma function.

Definition 2[21] The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x - \xi)^{n-\alpha-1} f(\xi) d\xi, \quad n-1 < \alpha < n \in \mathbb{N}.$$

Definition 3[2] The Caputo fractional derivative of order $\alpha > 0$ is defined by

$$D_*^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x - \xi)^{n-\alpha-1} \frac{d^n}{d\xi^n} f(\xi) d\xi, \quad n-1 < \alpha < n \in \mathbb{N}.$$

Definition 4[20] Let $\alpha \in \mathbb{R}^+$, $\Omega \subset \mathbb{R}$ an interval such that $a \in \Omega$, $a \leq x$, $x \in \Omega$. Then the following set of functions are defined:

$${}_a I_\alpha = \{ f \in C(\Omega) : I^\alpha f(x) \text{ exists and is finite in } \Omega \},$$

$${}_a D_\alpha = \{ f \in C(\Omega) : D_*^\alpha f(x) \text{ exists and is finite in } \Omega \}.$$

In view of these definitions we can conclude the following theorem:

Theorem 1 [20] Let $\alpha \in (0,1]$, $p \in \mathbb{N}$ and $f(x)$ a continuous function in $[a,b]$ satisfying the following conditions:

- (1) $D_*^{m\alpha} f \in C([a,b])$ and $D_*^{m\alpha} f \in {}_a I_\alpha([a,b])$, $\forall m = 1, 2, \dots, p$.
- (2) $D_*^{(p+1)\alpha} f(x)$ is continuous on $[a,b]$.

Then for each $x \in [a,b]$,

$$f(x) = \sum_{m=0}^p D_*^{m\alpha} f(a) \frac{(x-a)^{m\alpha}}{\Gamma(m\alpha+1)} + R_p(x,a),$$

$$\text{with } R_p(x,a) = D_*^{(p+1)\alpha} f(\xi) \frac{(x-a)^{(p+1)\alpha}}{\Gamma((p+1)\alpha+1)}, \quad a \leq \xi \leq x.$$

Remark 1 For simplicity we will use the operator D instead of D_* from now on.

3. Description of the Method

Given the mesh points, $\Delta: 0 = x_0 < x_1 < \dots < x_n = 1$ with $x_{k+1} - x_k = h$, $k = 0, 1, \dots, n-1$, and real numbers $\left\{ f_k, D^{\frac{1}{2}} f_k, \left(D^{\frac{1}{2}} \right)^4 f_k \right\}_{k=0}^n$ associated with the knots. We are going to construct spline interpolant S_Δ for which $D^{m\frac{1}{2}} S_\Delta(x_i) = D^{m\frac{1}{2}} f_i$, $i = 0, 1, \dots, n$, and $m=0, 1, 4$. This construction is given in the following two cases:

Case 1:

In this case, we suppose that the conditions of Theorem 1 are satisfied with $p = 4$, and $\alpha = 0.5$. Then we can define the spline interpolant as follows:

$$S_\Delta = S_k(x) = y_k + D^{\frac{1}{2}} y_k \frac{2(x-x_k)^{\frac{1}{2}}}{\sqrt{\pi}} + a_k(x-x_k) + b_k \frac{4(x-x_k)^{\frac{3}{2}}}{3\sqrt{\pi}} + \left(D^{\frac{1}{2}} \right)^4 y_k \frac{(x-x_k)^2}{2}, \quad (1)$$

where $x_k \leq x \leq x_{k+1}$ and $k = 0, 1, \dots, n-1$.

4 Existence and Uniqueness

If we require that $S_\Delta(x)$ and $D^{\frac{1}{2}} S_\Delta(x)$ is continuous on $[0,1]$, then it is easy to prove that formula (1) exists and is unique. That is, clear from the continuity conditions of $S_\Delta(x)$ and $D^{\frac{1}{2}} S_\Delta(x)$ from which we get:

$$y_{k+1} = y_k + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} D^{\frac{1}{2}} y_k + h a_k + b_k \frac{4h^{\frac{3}{2}}}{3\sqrt{\pi}} + \frac{h^2}{2} \left(D^{\frac{1}{2}} \right)^4 y_k, \quad (2)$$

and

$$D^{\frac{1}{2}} y_{k+1} = D^{\frac{1}{2}} y_k + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} a_k + h b_k + \frac{4h^{\frac{3}{2}}}{3\sqrt{\pi}} \left(D^{\frac{1}{2}} \right)^4 y_k. \quad (3)$$

Let

$$A_k = y_{k+1} - y_k - \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} D^{\frac{1}{2}} y_k - \frac{h^2}{2} \left(D^{\frac{1}{2}} \right)^4 y_k,$$

and let

$$B_k = D^{\frac{1}{2}} y_{k+1} - D^{\frac{1}{2}} y_k - \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} \left(D^{\frac{1}{2}} \right)^4 y_k.$$

Then the equations of (2) and (3), respectively, become

$$A_k = h a_k + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} b_k, \quad (4)$$

$$B_k = \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} a_k + h b_k. \quad (5)$$