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Numerical solution of two-dimensional nonlinear Volterra integral equations using Bernstein polynomials

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Abstract. The purpose of this paper is to present a numerical method for finding an approximate solution of two-dimensional (2D) nonlinear volterra integral equations. First, we introduce two-dimensional Bernstein functions, then present their operational matrices of integration and product. Using this properties and collcation points, reduce integral equation to a system of nonlinear algebric equations. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique.

Keywords: Two-dimensional integral equations, Bernstein polynomials, Operational matrix of integration, Product operational matrix,Volterra, Nonlinear.

1. Introduction

Mathematical modeling of real-life problems usually results in functional equations, e.g. integral equations, integro-differential equations and others. Inparticular, integral and integro-differential equations arise in fluid dynamics, biological models and chemical kinetics. Finding the analytical solutions of mentioned equations is not possible, and thus numerical methods are required [1,2,3,4,5].

In recent years, researchers have allocated considerable effort to study of numerical solutions of the two-dimensional integral and integro-differential equations. Many powerful methods have been proposed. In [6] authors, have applied rationalized Haar functions to the solution of the two-dimensional nonlinear integral equations.While in [7] Legendre polynomials have been chosen. In [8] triangular functions have been used by the authors and Block-pulse functions have been chosen by the authors of [9, 10]. In [11] the authors, have applied the differential transform method to solve two-dimensional volterra integral equations. The Euler method have been used in [12] to approximate the solution of 2D Volterra integral equations numerically by authors.

Consider the second-kind Volterra integral equation [7]

$$\begin{aligned} U(x,t) = \int_0^t\int_0^x k_1(x,t,y,z)\,H(y,z,U(y,z))\,dydz + \int_0^x k_2(x,t,y)\,G(y,t,U(y,t))\,dy \\ + \int_0^t k_3(x,t,z)\,F(x,z,U(x,z))\,dz + R(x,t), \qquad (x,t)\in[0,1]\times[0,1], \end{aligned} \tag{1}$$

where $U(x,t)$ is the unknown function in $\Omega\,(\Omega=[0,1]\times[0,1])$, the functions R, k_1, k_2 and k_3 are given smooth functions and the functions H, G and F are given continuos functions in $\Omega\times(-\infty,\infty)$, nonlinear in U.

As shown in [12], Eq. (1) arise from the transformation of certain Volterra integral equations of the first kind. In this paper, the numerical solution of Eq. (1) is computed by using 2D Bernstein polynomials.

The basis in the present method is the use operational matrices of the Bernstein polynomials. Using Bernstein polynomials expanded in terms of Legendre basis are given the mentioned operational matrices. The main reason to use this extension is reduce the computational, In particular operational matrices calculation.

This paper is organized as follows. In section 2, we introduce 2D Bernstein functions, their propertice

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and present operational matrices of them. In section 3, how the collocation method can be used to reduce the problem to a system of nonlinear equations is explained. Numerical examples are given in section 4 to evaluation of our method and comparison with the numerical results obtained by other authors is provided. Finally, conclusions are given in section 5.

2. Properties of 2D Bernstein polynomials

2.1 Definition and function approximation

Two-dimensional Bernstein functions are defined on Ω as

$$B_{i,j,M,N}(x,t) = B_{i,M}(x)\ B_{j,N}(t), \qquad i = 0,...,M,\ \ j = 0,..., N,$$

here $B_{i,M}$ and $B_{j,N}$ are the well-known Bernstein functions respectively of degree M and N, which are defined on the interval [0, 1] and can be determined with the following formula [13]:

$$B_{i,M}(x) = \binom{M}{i} x^i (1-x)^{M-i}, \qquad i = 0, ..., M,$$

where

$$\binom{M}{i} = \frac{M!}{i!(M-i)!}\ .$$

2D Bernstein polynomials form a single partition on Ω as:

$$\sum_{i=0}^{M}\sum_{j=0}^{N} B_{i,j,M,N}(x,\ t) = 1, \qquad (x,t) \in \Omega.$$

Suppose that $X = L^2(\Omega)$, the inner product in this space is defined by

$$\langle\ f(x,t),\ g(x,t)\rangle = \int_0^1\int_0^1 f(x,t)\,g(x,t)\,dxdt\ ,$$

and the norm is as:

$$\left\| f(x,t) \right\|_2 = \langle\ f(x,t),\ f(x,t)\rangle^{\frac{1}{2}} = (\int_0^1\int_0^1 |\ f(x,t)\,|^2\ dxdt)^{\frac{1}{2}}.$$

Let

$$\{B_{00}(x,t),\ldots,B_{0N}(x,t),\ldots,B_{M0}(x,t),\ldots,B_{MN}(x,t)\} \subset X\ ,$$

be the set of 2D Bernstein functions and

$$X_{M,N} = span\{B_{00}(x,t),\ \ldots,\ B_{0N}(x,t),\ \ldots,\ B_{M0}(x,t),\ \ldots,\ B_{MN}(x,t)\},$$

and $f(x,t)$ be an arbitrary function in X. Since $X_{M,N}$ is a finite dimensional vector space, f has a unique best approximation $f_{M,N} \in X_{M,N}$ [7], such that

$$f(x,t) \cong f_{M,N}(x,t) = F^T B(x,t) = \sum_{i=0}^{M}\sum_{j=0}^{N} f_{ij}\ B_{i,j,M,N}(x,t) = \sum_{k=0}^{M}\sum_{h=0}^{N} c_{kh}\ L_{kh}(x,t), \tag{2}$$

where $L_{kh}(x,t),\ k = 0,\ \ldots,\ M,\ h = 0,\ \ldots,\ N$ are 2D shifted Legendre functions on Ω [7] and coefficients c_{kh} are obtained by

$$c_{kh} = \frac{\langle\ f(x,t), L_{kh}(x,t)\rangle}{\left\|L_{kh}\ (\mathrm{x,t})\right\|_2^2}\ .$$

2D Legendre polynomial $L_{mn}(x,t)$ can be expanded in terms of the Bernstein basis as follows: