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On the Growth Estimate of Iterated Entire Functions

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Abstract. In this paper we study some growth properties of iterated entire functions which improve some earlier results.

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1. Introduction

Let $f(z)$ and $g(z)$ be two transcendental entire functions defined in the open complex plane **C**. It is well known [1], {[15], p-67, Th-1.46} that $\lim_{r\to\infty}\dfrac{T(r,fog)}{T(r,f)}=\infty$ and $\lim_{r\to\infty}\dfrac{T(r,fog)}{T(r,g)}=\infty$.

After this Singh [12], Lahiri [8], Song and Yang [14], Singh and Baloria [13], Lahiri and Sharma [9] and Datta and Biswas [3], [4] proved different results on comparative growth property of composite entire functions.

In this paper, we investigate the comparative growth of iterated entire functions in terms of its (p,q)-th order which is the generalization of previous results. We do not explain the standard notations and definitions of the theory of entire functions as those are available in [5], [15] and [16].

The following definitions are well known.

Definition 1.1. The orde ρ_f and lower order λ_f of a meromorphic function $f(z)$ is defined as

$$\rho_f=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}\quad\text{and}\quad \lambda_f=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

If $f(z)$ is entire then

$$\rho_f=\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r}\quad\text{and}\quad \lambda_f=\liminf_{r\to\infty}\frac{\log\log M(r,f)}{\log r}.$$

Notation 1.2. [11] $\log^{[0]}x=x$, $\exp^{[0]}x=x$ and for positive integer m, $\log^{[m]}x=\log(\log^{[m-1]}x)$, $\exp^{[m]}x=\exp(\exp^{[m-1]}x)$.

Definition 1.3. [6] The (p,q)-th order $\rho_f(p,q)$ and lower (p,q)-th order $\lambda_f(p,q)$ of a meromorphic function $f(z)$ is define as

$$\rho_f(p,q)=\limsup_{r\to\infty}\frac{\log^{[p-1]}T(r,f)}{\log^{[q]}r}\quad\text{and}\quad \lambda_f(p,q)=\liminf_{r\to\infty}\frac{\log^{[p-1]}T(r,f)}{\log^{[q]}r}.$$

If $f(z)$ is entire then

$$\rho_f(p,q)=\limsup_{r\to\infty}\frac{\log^{[p]}M(r,f)}{\log^{[q]}r}\quad\text{and}\quad \lambda_f(p,q)=\liminf_{r\to\infty}\frac{\log^{[p]}M(r,f)}{\log^{[q]}r}.$$

where $p>q$. Clearly $\rho_f(2,1)=\rho_f$ and $\lambda_f(2,1)=\lambda_f$.

According to Lahiri and Banerjee [7] if $f(z)$ and $g(z)$ are entire functions then the iteration of $f(z)$ with respect to $g(z)$ is defined as follows:

$$
\begin{aligned}
f_1(z) &= f(z)\\
f_2(z) &= f(g(z)) = f(g_1(z))\\
f_3(z) &= f(g(f(z))) = f(g_2(z)) = f(g(f_1(z)))\\
&\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots\\
f_n(z) &= f(g(f \ldots\ldots\ldots\ldots(f(z) \text{ or } g(z)) \ldots\ldots\ldots))\\
&\qquad \text{according as } n \text{ is odd or even,}
\end{aligned}
$$

and so

$$
\begin{aligned}
g_1(z) &= g(z)\\
g_2(z) &= g(f(z)) = g(f_1(z))\\
g_3(z) &= g(f(g(z))) = g(f_2(z)) = g(f(g_1(z)))\\
&\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots\\
g_n(z) &= g(f_{n-1}(z)) = g(f(g_{n-2}(z))).
\end{aligned}
$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [5] Let $f(z)$ be an entire function. For $0 \le r < R < \infty$, we have

$$T(r, f) \le \log^+ M(r, f) \le \frac{R+r}{R-r} T(R, f).$$

Lemma 2.2. [1] If $f(z)$ and $g(z)$ are any two entire functions, for all sufficiently large values of *r*,

$$M\left(\frac{1}{8} M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \le M(r, fog) \le M(M(r, g), f).$$

Lemma 2.3. [10] Let $f(z)$ and $g(z)$ be two entire functions. Then we have

$$T(r, fog) \ge \frac{1}{3} \log M\left(\frac{1}{8} M\left(\frac{r}{4}, g\right) + O(1), f\right).$$

Lemma 2.4. Let $f(z)$ and $g(z)$ be two entire functions of non- zero finite (p,q)-th order $\rho_f(p,q)$ and $\rho_g(p,q)$ respectively, then for any $\varepsilon > 0$ and $p > q$,

$$\log^{[(n-1)p-(n-2)q]} M(r, f_n) \le \begin{cases} (\rho_f(p,q) + \varepsilon) \log^{[q]} M(r, g) + O(1) & \text{when } n \text{ is even,} \\ (\rho_g(p,q) + \varepsilon) \log^{[q]} M(r, f) + O(1) & \text{when } n \text{ is odd} \end{cases}$$

for all sufficiently large values of *r*.

Proof. First suppose that *n* is even. Then from second part of Lemma 2.2 and Definition of (p,q)-th order, it follows that for all sufficiently large values of *r*,

$$
\begin{aligned}
& M(r, f_n) \le M(M(r, g_{n-1}), f)\\
\text{i.e., } & \log^{[p]} M(r, f_n) \le \log^{[p]} M(M(r, g_{n-1}), f)\\
& \qquad \le (\rho_f(p,q) + \varepsilon) \log^{[q]} M(r, g_{n-1})\\
\text{So, } & \log^{[p+1]} M(r, f_n) \le \log^{[q+1]} M(r, g(f_{n-2})) + O(1)\\
\text{i.e., } & \log^{[p+1-q]} M(r, f_n) \le \log M(r, g(f_{n-2})) + O(1).
\end{aligned}
$$

Taking repeated logarithms *p-1* times, we get