

Numerical Solution of Two-dimensional Nonlinear Volterra Integro-differential Equations by Tau Method

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Abstract. In this paper, a method is employed to approximate the solution of two-dimensional nonlinear Volterra integro-differential equations (2DNVIDEs) with supplementary conditions. First, we introduce two-dimensional Legendre polynomials, then convert 2DNVIDEs to the two-dimensional linear Volterra integro-differential equations (2DLVIDEs). Using this properties and collocation points, reduce it to the system of algebraic equations. Finally, some numerical examples are given to clarify the efficiency and accuracy of the present method.

Keywords: Two dimensional, Linear, Nonlinear, Volterra integro-differential equation, Tau method .

1. Introduction

Generally, real-world physical problems are modelled as differential, integral and integro-differential equations. Since finding the solution of these equations is too complicated, in recent years a lot of attention has been devoted by researchers to find the analytical and numerical solution of these equations. In [1] authors, have applied Legendre polynomials to solve two-dimensional Volterra integral equations. While in [2] Homotopy Perturbation and Differential transform methods have been chosen. Iterative methods have been used in [3] to solve two-dimensional nonlinear Volterra-Fredholm integro-differential equations. In [4] the authors, have applied the differential transform method for the system of two-dimensional nonlinear Volterra integro-differential equations. The Tau method has been developed for the numerical solution of two-dimensional linear Volterra integro-differential equations in [5] and Block-pulse functions have been used for solve two-dimensional Volterra integro-differential equations by the authors of [6].

On the other hand, 2DNVIDEs have interesting applications in Physics, Mechanics and applied sciences. For this reason, in this paper, we obtained numerical solution of two-dimensional nonlinear Volterra integro-differential equations with given supplementary conditions. To this end, we replace the differential and integral parts of 2DLVIDEs by Legendre polynomials and then convert it to a corresponding system of algebraic equations. In a similar manner, we transform the supplementary conditions to a algebraic system of equations. Finally, by combining these two systems of algebraic equations, we obtain a system of linear algebraic equations and solve it to obtain an approximate solution of the problem.

This paper is organized as follows. In Section 2, we describe properties of Legendre polynomials. In section 3, we explain numerical solution 2DLVIDEs by Tau method. Numerical examples are given in section 4 to evaluation of our method. Finally, conclusions are given in section 5.

2. Properties of Legendre polynomials

2.1. Definition the Legendre functions

The Legendre polynomials are defined as [7]:

$$L_0(x) = 1, \quad L_1(x) = x, \quad x \in [-1, 1],$$

$$L_i(x) = \left(2 - \frac{1}{i}\right)x L_{i-1}(x) - \left(1 - \frac{1}{i}\right)L_{i-2}(x) \quad i = 2, 3, 4, \dots$$

So the shifted Legendre polynomials are defined as:

$$L_0^*(x) = 1, \quad L_1^*(x) = \frac{2x-l}{l}, \quad x \in [0, l],$$

for $i \geq 2$ as

$$L_i^*(x) = (2 - \frac{1}{i})(\frac{2x-l}{l})L_{i-1}^*(x) - (1 - \frac{1}{i})L_{i-2}^*(x),$$

with the orthogonally condition as:

$$\int_0^l L_i(t) L_j(t) dt = \begin{cases} \frac{l}{2i+1}, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

2D shifted Legendre functions are defined on Ω ($(x, t) \in \Omega = [0, l] \times [0, T]$, l, T are finite constants) as [8]:

$$\psi_{mn}(x, t) = L_m(\frac{2}{l}x - 1)L_n(\frac{2}{T}t - 1), \quad m, n = 0, 1, 2, 3, \dots \quad (1)$$

here L_m, L_n are the well-known Legendre functions respectively of order m and n .

2D shifted Legendre functions $\psi_{mn}(x, t)$ are orthogonal with each other as:

$$\int_0^T \int_0^l \psi_{ij}(x, t) \psi_{mn}(x, t) dx dt = \begin{cases} \frac{lT}{(2m+1)(2n+1)}, & i = m, j = n, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $X = L^2(\Omega)$, the inner product in this space is defined by

$$\langle f, g \rangle = \int_0^T \int_0^l f(x, t) g(x, t) dx dt, \quad (2)$$

and the norm is as follows:

$$\|f\|_2 = \langle f, f \rangle^{\frac{1}{2}}.$$

Let

$$\{\psi_{00}(x, t), \psi_{01}(x, t), \dots, \psi_{0N}(x, t), \dots, \psi_{N0}(x, t), \dots, \psi_{NN}(x, t)\} \subset X,$$

be the set of 2D shifted Legendre functions and

$$X_{NN} = \text{span}\{\psi_{00}(x, t), \psi_{01}(x, t), \dots, \psi_{0N}(x, t), \dots, \psi_{N0}(x, t), \psi_{N1}(x, t), \dots, \psi_{NN}(x, t)\},$$

and $f(x, t)$ be an arbitrary function in X . Since X_{NN} is a finite dimensional vector space, f has a unique best approximation $f_{NN} \in X_{NN}$ [9], such that

$$\forall g \in X_{NN}, \|f - f_{NN}\|_2 \leq \|f - g\|_2.$$

Moreover since $f_{NN} \in X_{NN}$ there exist unique coefficients $f_{00}, f_{01}, \dots, f_{NN}$ such that

$$f(x, t) \cong f_{NN}(x, t) = \sum_{i=0}^N \sum_{j=0}^N f_{ij} \psi_{ij}(x, t) = F^T \psi(x, t) = \psi^T(x, t) F,$$

where F and $\psi(x, t)$ are $(N+1) \times (N+1)$ vectors with the form

$$F = [f_{00}, \dots, f_{0N}, f_{10}, \dots, f_{1N}, \dots, f_{N0}, \dots, f_{NN}]^T,$$

$$\psi(x, t) = [\psi_{00}(x, t), \dots, \psi_{0N}(x, t), \psi_{10}(x, t), \dots, \psi_{1N}(x, t), \dots, \psi_{N0}(x, t), \dots, \psi_{NN}(x, t)]^T,$$

2D shifted Legendre function coefficients f_{NN} are obtained by

$$f_{mn} = \frac{\langle f(x, t), \psi_{mn}(x, t) \rangle}{\|\psi_{mn}(x, t)\|_2^2} = \frac{(2m+1)(2n+1)}{lT} \int_0^T \int_0^l f(x, t) \psi_{mn}(x, t) dx dt.$$

Similarly, any function $k(x, t, y, z) \in L^2(\Omega \times \Omega)$, can be expanded in terms of 2D shifted Legendre functions as

$$k(x, t, y, z) \cong \psi^T(x, t) K \psi(y, z), \quad (3)$$