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The Iterative Method for Optimal Control Problems by the Shifted Legendre Polynomials

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Abstract. In this paper an iterative method based on shifted Legendre polynomials is presented to obtain the approximate solutions of optimal control problems subject to integral equations. The operational matrices of integration and product of shifted Legendre polynomials for solving integral equation is employed. The methodology is based on the parametrization of control and state functions. This converted the problem to nonlinear optimization problem in any iteration. In addition, some numerical examples are presented to illustrate the accuracy and efficiency of the proposed method.

Keywords: Optimal control problem, Legendre polynomials, Iterative method.

1. Introduction

The classical theory of optimal control was developed in the last years as a powerful tool to create optimal solutions for real processes in many aspects of science and technology. Complexity of applying analytical methods for obtaining fast and near optimal solutions is the reason for creating numerical approaches. The numerical methods for solving optimal control problems described by ODE or integral equations can be found in [1]. Some interesting iterative schemes with their convergence for optimal control of Volterra integral equations considering some conditions on the kernel of integral equation was introduced in [2-4]. The method of parametrization for solving some classes of optimal control problems were proposed [5-7]. Lee and Chang [8] appeared to be the first to study optimal control problem of nonlinear systems using general orthogonal polynomials. Chebyshev polynomials [9] were used for solving nonlinear optimal control problem. A Fourier based state parameterizations approach for solving linear quadratic optimal control problems proposed in [10]. The hybrid functions consisting of the combination of block-pulse functions with Chebyshev polynomials [11], Legendre polynomials [12] and Taylor series [13, 14] shown to be a mathematical power tool for discretization of selected problems. Also some methods based on approximating the Volterra integral equation can be seen in [15, 16]. In this paper, we considered the numerical solution of a class of optimal control problems subject to integral equations, which is described by the following minimizing problem:

$$J(x,u)=\int_0^1 f\left(t,x(t),u(t)\right)dt, \tag{1}$$

subject to:

$$\mathrm{x(t)}=y(t)+\lambda_1\int_0^t K_1\left(t,s,x(s),u(s)\right)ds++\lambda_2\int_0^1 K_2\left(t,s,u(s)\right)x(s)ds, \tag{2}$$

where

$\lambda_1,\lambda_2\in\mathbb{R}$, $f\in C\left([0,1]\times\mathbb{R}\times\mathbb{R}\right)$ and $y(.)\in C^{+\infty}\left([0,1]\right)$ are given function. $x(.),\, u(.)\in C^{+\infty}\left([0,1]\right)$ are the state and control functions, respectively, which to be determined and the given kernel functions, $K_1(t,s,x(s),u(s))$ is smooth in $C^{+\infty}\left([0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\right)$ and $K_2(t,s,u(s))$ is smooth in $C^{+\infty}\left([0,1]\right)\times C^{+\infty}\left([0,1]\right)\times\mathbb{R}$. Here, we assume that the problem (1)- (2) has a unique solution. Due to the absence of an approximate numerical method for solving this kind of optimal control problems the main purpose of this article is to present a direct numerical method for obtaining approximate solutions of the Eqs.

(1) and (2) by using parametrization and shifted Legendre polynomials. The paper is organized as follows: in section 2, the shifted Legendre polynomials and their properties are presented. Section 3 is devoted to the solution method. In section 4, we reported our numerical findings and demonstrated the accuracy of the proposed method.

2. Properties of shifted Legendre polynomials

A set of shifted Legendre polynomials, denote by $\{L_k(t)\}$ for $k=0,1,2,\cdots$ is orthogonal with respect to the weighting function $w(t)=1$, over interval $[0,1]$, can be generated from the recurrence relation [17]

$$(\mathrm{k}+1)\,\mathrm{L}_{k+1}(t)=(2\,\mathrm{k}+1)(2\,\mathrm{t}-1)\,\mathrm{L}_k(t)-kL_{k-1}(t),\quad k=1,2,3,\cdots \tag{3}$$

With

$$L_0(t)=1,\quad L_1(t)=2t-1. \tag{4}$$

The orthogonality of these polynomials is expressed by the relation

$$\int_0^1 L_j(\mathrm{t})L_k(t)dt=\begin{cases}\dfrac{1}{2k+1}, & j=k\\ 0, & j\neq k\end{cases} \tag{5}$$

A function, $f(t)$ square integrable in $[0,1]$, may be expressed in terms of shifted Legendre polynomials as

$$f(t)\approx P_m(t)=\sum_{i=0}^{m}c_iL_i(t),\qquad m\geq 0 \tag{6}$$

Equation (6) can be written as

$$P_m(t)=C^TL(t) \tag{7}$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $L(t)$ are given by

$$C=\left[c_0,c_1,c_2,\cdots,c_m\right]^T, \tag{8}$$

$$L(t)=\left[L_0(t),L_1(t),L_2(t),\cdots,L_m(t)\right]. \tag{9}$$

The use of an orthogonal basis on $[0,1]$ allows us to directly obtain the least-squares coefficients of $P_m(t)$ in that basis, and also ensure permanence of these coefficients with respect to the degree m the approximate, that is, all coefficients of $P_{m+1}(t)$ agree with those of $P_m(t)$, except for that of the newly introduced term. By using Eq. (5) the Legendre coefficients are given by

$$c_j=(2j+1)\int_0^1 L_j(t)\,f(t)\,\mathrm{dt},\qquad \mathrm{j}=0,1,2,\cdots,\mathrm{m}. \tag{10}$$

Also, the integration of the cross, product of two vector $L(t)$ in Eq. (9) is

$$D=\int_0^1 L(t)L^T(t)\,\mathrm{dt}=diag\left(1,\frac{1}{3},\cdots,\frac{1}{2m+1}\right), \tag{11}$$

where, D is the $(m+1)\times(m+1)$ diagonal matrix.

The integrating of the vector $L(t)$ defined in Eq. (9) is given by

$$\int_0^1 L(x)\,\mathrm{d}\,x\approx PL(t), \tag{12}$$

where P is the $(m+1)\times(m+1)$ operational matrix of integration of the shifted Legendre polynomials is given by [18], and it is a tridiagonal matrix.