

Numerical Treatment of Lacunary Spline Function of Fractional Order

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Abstract. In this paper, effects of lacunary fractional derivatives and existence/ uniqueness on the fifth order spline function in fractional order have been studied. These results use in the convergence analysis of fractional spline polynomial method for such problems numerically. Theoretical results are illustrated with efficient of the absolute errors by some fractional differential examples.

Keywords: caputo derivatives, spline function, convergence analysis.

1. Introduction

Fractional calculus is a field of mathematics arise in various areas of science and engineering. In the recent past years, that grows of the traditional definitions of the integral and derivative theory, based on these requirements, the numerical analysis has become very important of fractional differential equations (see, [6, 8, 12, 13, 14]).

Interpolation polynomials are generalized of approximating solution of data interpolated functions and ordinary differential equations to an arbitrary order. These approximations have attracted considerable interest because of their ability to solving complex phenomena. In recent years the splines of fractional order have also had a significant increasing attention of the wavelet transform [5, 10]. We refer to the recent papers of describe properties fractional spline derivatives that make them useful in a variety of applications [7, 9, 11].

In this paper, we will extend the construction of fractional quintic spline polynomial on (faraidun& pshtiwan [4]). We prove the existence and uniqueness of the model of spline function in Theorem 1, also prove the convergence of the method and drive error estimates for the all derivatives gives effective for fractional derivatives of the method. Finally, the theoretical results illustrated by some numerical examples and the absolute errors be more efficient with good accuracy when compared to [11].

2. System description

We start the definitions of the fractional Taylor series and Caputo fractional derivative and integral:

Definition 2.1 (Sabatier and Agrawal, 2007, Michael and B. Thierry, 2000, [3, 10,] respectively) The Caputo fractional derivative of order $\alpha > 0$, is defined by

$$D_a^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(s)}{(x-s)^{\alpha+1-n}} ds & \text{for } n-1 < \alpha < n, n \in \mathbb{N} \\ \frac{d^n}{dx^n} f(x) & \text{for } \alpha = n, n \in \mathbb{N} \end{cases}$$

Definition 2.2 (Podlubny, 1999 [2])

Suppose that $D_a^{k\alpha} f(x) \in C[a, b]$ for $k = 0, 1, \dots, n+1$ where $0 < \alpha \leq 1$, then we have the Taylor Series expansion about $x = \tau$,

$$f(x) = \sum_{i=0}^n \frac{(x-\tau)^{i\alpha}}{\Gamma(i\alpha+1)} D_a^{i\alpha} f(\tau) + \frac{(D_a^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-\tau)^{(n+1)\alpha}$$

with $a \leq \xi \leq x$, for all $x \in (a, b]$, where $D_a^{k\alpha} = D_a^\alpha \cdot D_a^\alpha \dots D_a^\alpha$ (k time).

Definition 2.3 (Kincaid and Cheney, 2002 [1])

The modulus of continuity of a function f continuous on a segment $[a, b]$, $f \in C[a, b]$ is a function $\omega(t) = \omega(f, t)$ defined for $t \in [0, b - a]$ by the relation $\omega(t) = \omega(f, t; \delta) = \max_{|t-x| \leq \delta} |f(t) - f(x)|$.

3. Description of the method

We will construct the spline function of fifth order with fractional lacunary derivatives in equation (1) are developed, for the difference degree polynomial with boundary conditions is appeared in (Faraidun and Pshtiwan, 2015, Fawzy and Saxena [4, 15, 16] respectively), and $S_\Delta(x)$ in the $C^5[a, b]$ that satisfy the boundary conditions as follows:

$$S(x_k) = y_k + \frac{2}{\sqrt{\pi}}(x-x_k)^{\frac{1}{2}}a_k + (x-x_k)Dy_k + \frac{4}{3\sqrt{\pi}}(x-x_k)^{\frac{3}{2}}b_k + \frac{(x-x_k)^2}{2}D^{(2)}y_k + \frac{8}{15\sqrt{\pi}}(x-x_k)^{\frac{5}{2}}c_k + \frac{(x-x_k)^3}{6}D^{(3)}y_k + \frac{16}{105\sqrt{\pi}}(x-x_k)^{\frac{7}{2}}d_k + \frac{(x-x_k)^4}{24}D^{(4)}y_k + \frac{32}{945\sqrt{\pi}}(x-x_k)^{\frac{9}{2}}D^{(5)}y_k + \frac{(x-x_k)^5}{120}D^{(5)}y_k \quad (1)$$

On the interval $[x_k, x_{k+1}]$ where a_k, b_k, c_k and $d_k, k = 0, 1, 2, 3, \dots, n-1$, are unknowns to be determined, with the boundary continuity conditions

$$S^{(j)}(x_k) = y_{k+1}^{(j)}, j=0, 1, 2, 3, 4, 9/2, 5.$$

Theorem 3.1: Let $S(x) \in C^5[a, b]$, $x \in [x_k, x_{k+1}]$, $k = 0, 1, 2, 3, \dots, n-1$, then there exists a unique spline function with fractional order for formula (1).

Proof: From the construct formula in equation (1), can be using the continuity conditions to obtain:

$$A_k = y_{k+1} - y_k - hDy_k - \frac{h^2}{2}D^{(2)}y_k - \frac{h^3}{6}D^{(3)}y_k - \frac{h^4}{24}D^{(4)}y_k - \frac{32}{945\sqrt{\pi}}h^{\frac{9}{2}}D^{(5)}y_k + \frac{h^5}{120}D^{(5)}y_k \quad (2)$$

where $A_k = \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}a_k + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}b_k + \frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}c_k + \frac{16}{105\sqrt{\pi}}h^{\frac{7}{2}}d_k$, $h = x - x_k$,

$$B_k = Dy_{k+1} - Dy_k - hD^{(2)}y_k - \frac{h^2}{2}D^{(3)}y_k - \frac{h^3}{6}D^{(4)}y_k - \frac{16}{105\sqrt{\pi}}h^{\frac{7}{2}}D^{(5)}y_k - \frac{h^4}{24}D^{(5)}y_k \quad (3)$$

where $B_k = \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}b_k + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}c_k + \frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}d_k$,

$$C_k = D^{(2)}y_{k+1} - D^{(2)}y_k - hD^{(3)}y_k - \frac{h^2}{2}D^{(4)}y_k - \frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}D^{(5)}y_k - \frac{h^3}{6}D^{(5)}y_k \quad (4)$$

where $C_k = \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}c_k + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}d_k$,

$$D_k = D^{(3)}y_{k+1} - D^{(3)}y_k - hD^{(4)}y_k - \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}D^{(5)}y_k - \frac{h^2}{2}D^{(5)}y_k \quad (5)$$

where $D_k = \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}d_k$.

Equating these equations (2) to (5), to obtain the following:

$$a_k = \frac{\sqrt{\pi}}{1890}h^{-\frac{1}{2}}[945A_k - 630hB_k + 168h^2C_k - 16h^3D_k], \quad (6)$$

$$b_k = \frac{\sqrt{\pi}}{90}h^{-\frac{1}{2}}[45B_k - 30hC_k + 8h^2D_k], \quad (7)$$

$$c_k = \frac{\sqrt{\pi}}{6}h^{-\frac{1}{2}}[3C_k - 2hD_k], \quad (8)$$

$$d_k = \frac{\sqrt{\pi}}{2}h^{-\frac{1}{2}}D_k. \quad (9)$$

Lemma 3.2: Let $y(x_k) \in C^5[a, b]$, and the error estimate of the spline function in equation (1), for the unknown derivatives is