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Periodic solutions for singular Liénard equations with indefinite weight

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Abstract In this paper, the problem of periodic solutions is studied for singular Liénard equations

$$\ddot{x}(t)+f(x(t))\dot{x}(t)+\varphi(t)x^{\mu}(t)=h(t),$$

where $f:(0,+\infty)\to R$ is continuous and has a singularity at the origin, μ is a positive constant. By using a continuation theorem of coincidence degree theory, a new result on the existence of positive periodic solutions is obtained. The interesting thing is that the sign of weight $\varphi(t)$ is allowed to change for $t\in[0,T]$.

Keywords: Liénard equation, Continuation theorem, Periodic solution

1. Introduction

In this paper, we are concerned with the existence of positive T −periodic solutions for the equations

$$\ddot{x}(t)+f(x(t))\dot{x}(t)+\varphi(t)x^{\mu}(t)=h(t)\,, \tag{1.1}$$

where $f\in C((0,+\infty),R)$, φ is T −Periodic function with φ in $L([0,T],R)$, μ is a positive constant. In this equation, the function $f(x)$ has a singularity at $x=0$, i.e., $\lim\limits_{x\to 0^+}f(x)=+\infty$. Besides this, the sign of $\varphi(t)$ is allowed to change. The equations of this type arise in modelling of important problems appearing in many physical contexts (see [1]-[5] and the references therein).

In the past years, under the conditions of $\varphi(t)\geq 0$ and $\alpha(t)\geq 0$ for a.e. $t\in[0,T]$, the problem of existence of periodic solutions to the equation without friction term

$$\ddot{x}(t)+\varphi(t)x(t)-\frac{\alpha(t)}{x^{\mu}}=h(t)$$

has been extensively studied by [6]-[10]. Beginning with the paper of Habets-Sanchez [11], many researchers in [12]-[15] have considered the classical Liénard equation with a singularity of repulsive type

$$\ddot{x}(t)+f(x(t))\dot{x}(t)+\varphi(t)x(t)-\frac{\alpha(t)}{x^{\mu}}=h(t).$$

In these papers, apart from the function $\varphi(t)$ satisfies $\varphi(t)\geq 0$ for a.e. $t\in[0,T]$, $f(x)$ being continuous on $[0,+\infty)$ is needed. For the recent development of this area, we refer readers to the literature [16]-[19]. But up to our knowledge, few papers have considered the case where $f(x)$ has a singularity at $x=0$, and the sign of $\varphi(t)$ is indefinite. The reason for this is that, in such situation, the equation may have no a priori estimates.

Throughout this paper, let $C_T=\{x\in C(R,R): x(t+T)=x(t) \text{ for all } t\in R\}$ with the norm defined by $|x|_\infty=\max\limits_{x\in[0,T]}|x(t)|$, and $C_T^1=\{x\in C^1(R,R): x(t+T)=x(t) \text{for all } t\in R\}$ with the norm defined by $\|x\|_{C_T}=\max\{|x|_\infty,|\dot{x}|_\infty\}$. For any $T-$periodic solution $y(t)$ with $y\in L([0,T],R)$, $y_+(t)$ and $y_-(t)$ is denoted by $\max\{y(t),0\}$ and $\min\{y(t),0\}$, respectively, and $\bar{y}=\frac{1}{T}\int_0^T y(s)ds$. Clearly, $y(t)=y_+(t)-y_-(t)$ for all $t\in R$, and $\bar{y}=\overline{y_+}-\overline{y_-}$.

2. Preliminary lemmas

Lemma 2.1. [20] Assume that there exist positive constants m_0, m_1 and M^* with $0<m_0<m_1$, such that the following conditions hold.

1. For any $\lambda\in(0,1]$, each possible positive T −periodic solution μ to the equation

$$\ddot{u}(t)+\lambda f(u(t))\dot{u}(t)+\lambda\varphi(t)u^{\mu}(t)=\lambda h(t) \tag{2.1}$$

satisfies the inequalities $m_0<u(t)<m_1$ and $|\dot{u}(t)|<M^*$, for all $t\in[0,T]$.

2. The inequality

$$(\bar{h}-\bar{\varphi}m_0^{\mu})(\bar{h}-\bar{\varphi}m_1^{\mu})<0$$

holds.

Then, equation (1.1) has at least one T −periodic solution u such that $m_0 < u(t) < m_1$ for all $t \in [0,T]$.

Lemma 2.2. Let $u:[0,\omega] \to R$ be an arbitrary absolutely continuous function with $u(0)=u(\omega)$. Then the inequality

$$(\max_{[0,T]} u(t) - \min_{[0,T]} u(t))^2 \le \frac{\omega}{4}\int_4^{\omega} |\dot{u}(s)|^2 ds$$

holds.

Now, we embed equation (1.1) into the following equations family with a parameter $\lambda \in (0,1]$

$$\ddot{x}(t)+\lambda f(x(t))\dot{x}(t)+\lambda\varphi(t)x^{\mu}(t)=\lambda h(t), \lambda \in (0,1].$$

Let

$D=\{x \in C_T^1: \ddot{x}(t)+\lambda f(x(t))\dot{x}(t)+\lambda\varphi(t)x^{\mu}(t)=\lambda h(t), \lambda \in (0,1]; x(t)>0, \forall t \in [0,T]\}$,

$$F(x)=\int_1^x f(s)ds, G(x)=F(x)+x^{\mu}T\overline{\varphi_-}, x \in (0,+\infty), \tag{2.2}$$

where f(x) and μ are determined in (1.1).

Lemma 2.3. Assume $\bar{\varphi}>0$, then for each $u \in D$, there are constants $\xi_1, \xi_2 \in [0,T]$ such that

$$u(\xi_1) \le \left(\frac{\bar{h}}{\bar{\varphi}}\right)^{\frac{1}{\mu}} := \eta \tag{2.3}$$

and

$$u(\xi_2) \ge \left(\frac{\bar{h}}{\overline{|\varphi|}}\right)^{\frac{1}{\mu}} := \eta_0. \tag{2.4}$$

Proof. Let $u \in D$, then

$$\ddot{u}(t)+\lambda f(u(t))\dot{u}(t)+\lambda\varphi(t)u^{\mu}(t)=\lambda h(t),$$

which together with the fact of $u(t)>0$ for all $t \in [0,T]$ gives

$$\frac{\ddot{u}(t)}{u^{\mu}(t)}+\frac{\lambda f(u(t))\dot{u}(t)}{u^{\mu}(t)}+\lambda\varphi(t)=\lambda h(t).$$

Integrating the above equality over the interval $[0,T]$, we obtain

$$\int_0^T \frac{\ddot{u}(t)}{u^{\mu}(t)}dt+\lambda\int_0^T \varphi(t)dt=\lambda\int_0^T \frac{h(t)}{u^{\mu}(t)}dt,$$

i. e.,

$$\int_0^T \frac{\ddot{u}(t)}{u^{\mu}(t)}dt+\lambda T\bar{\varphi}=\lambda\int_0^T \frac{h(t)}{u^{\mu}(t)}dt.$$

Since the inequality

$$\int_0^T \frac{\ddot{u}(t)}{u^{\mu}(t)}dt \ge 0$$

is easily obtained by a simple integration by parts, it follows from (2.1) that

$$T\bar{\varphi} \le \int_0^T \frac{h(t)}{u^{\mu}(t)}dt=\frac{T\bar{h}}{u^{\mu}(\xi_1)}.$$

By using mean value theorem of integrals, we have that there exists a point $\eta \in [0,T]$ such that

$$T\bar{\varphi} \le \frac{T\bar{h}}{u^{\mu}(\xi_1)},$$

i. e.,