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An inverse problem for diffusive logistic equation with free boundary

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Abstract: This paper considers an inverse problem for a logistic model with free boundary. This inverse problem aims to identify the growth coefficient only depending on time from a fixed point measurement data. Based on a fixed point argument, we prove the local in time existence and uniqueness of our inverse problem.

Key words: diffusive logistic model; inverse problem; free boundary

1 Introduction

Free boundary problems are a kind of mathematical physics models with one unknown function that defines this boundary. It has been found in a broad variety of physical applications, such as the one-phase Stefan problem [18,15,3,14,1,6], the free boundary problems for predator-prey model [13], the information diffusion in online social networks with time varying distance [20], ductal carcinoma in situ mathematical model [21] and so on. In the last twenty years, there has been a few of work related to the related inverse problems, see [7,9,10,13] and the references therein for more details. Such inverse problems are more complicated than the traditional ones because of the unknown free boundary.

In this paper, we consider the following diffusive logistic model with free boundary [11]:

$$\begin{cases} u_t - du_{xx} = r(t)u\left(1-\frac{u}{K}\right), & (x,t)\in Q_{s,T}, \\ u_x(0,t)=u(s(t),t)=0, & t\in(0,T), \\ s(0)=s_0>0, & \\ u(x,0)=u_0(x), & x\in[0,s_0], \\ s'(t)=-\mu u_x(s(t),t), & t\in(0,T), \end{cases} \tag{1.1}$$

where $Q_{s,T}=\{(x,t)|0<x<s(t),0<t<T\}$, $s(t)$ represents the free boundary is unknown function. System (1.1) can be used to depict information diffusion in online social networks, in which $u(x,t)$ denotes the density of influenced users at time t and distance x, K and d indicate the carrying capacity and diffusion rate, respectively.

In this paper we couple to the equations the following additional the boundary observation on u:

$$u(0,t)=f(t) \quad t\in[0,T], \tag{1.2}$$

as our inversion input data to determine the unknown function $r(t)$, which represents the intrinsic growth rate in this model.

In [5], the authors proved showed the local in time existence and uniqueness of logistic model and further obtained blow-up property about a free boundary model. The authors [11] proved a global existence for a logistic equation with free boundary.

The last boundary condition $s'(t)=-\mu u_x(s(t),t)$ on boundary $s(t)$ is called Stefan condition, which is widely used to describe phase transitions between solid and fluid states [2].

Recently, inverse source problems with a free boundary have received much attention. For example, Snitko [11] proved the local in time existence and uniqueness for an inverse problem of determining an unknown time-dependent leading coefficient in a parabolic equation with free boundary. Hussein, Lesnic, Ivanchov and Snitko [8] investigated a multiple time-dependent coefficient identification thermal problem with unknown free boundary under two additional integral conditions.

In this paper, we consider a coefficient inverse problem for system (1.1), which is a semi-linear model with free boundary. On the other hand, we use the measurement at boundary point $x=0$. In practical applications our measurement data are less than the global measurement data. In this paper we will prove the local existence and uniqueness for our coefficient inverse problem of determining $r(t)$ in (1.1) by the measurement data (1.2).

The rest of our paper is organized as follows. In Section 2, we prove a local in time existence and uniqueness result for the direct free boundary problem and show the solution in suitable Sobolev space continuous dependence on T and r. In Section 3, we first transfer our inverse problem to an equivalent problem. Then a local existence and uniqueness of the equivalent problem is obtained by the contraction mapping.

2 Direct free boundary problem

In this section, we prove the existence local in time of the direct problem (1.1) in a suitable Banach space. Meanwhile, we show a continuous property of the solution with respect to r and T, which is important to consider the inverse problem of determining the unknown r.

Firstly, we make a change of variable to straighten the free boundary. Let $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, and

$$\xi = \frac{x}{s(t)}, \qquad u(x,t) = v(\xi,t), \tag{2.1}$$

system (1.1) can be rewritten as

$$\begin{cases} v_t - d\frac{1}{s^2(t)}v_{\xi\xi} - \frac{s'(t)}{s(t)}\xi v_\xi = r(t)v\left(1-\frac{v}{K}\right), & (\xi,t)\in Q_T, \\ v_\xi(0,t) = v(1,t) = 0, & t\in(0,T), \\ v(\xi,0) = v_0(\xi), & \xi \in \overline{\Omega}, \\ s(t)s'(t) = -\mu v_\xi(1,t), & t\in(0,T), \end{cases} \tag{2.2}$$

where $v_0(\xi) = u_0(x)$, and (1.2) is rewritten as

$$v(0,t) = f(t), \qquad t\in[0,T] \tag{2.3}$$

Let

$$h(t) = s(t)s'(t). \tag{2.4}$$

Then, (v,h) further satisfies the following problem:

$$\begin{cases} v_t - dA(h)v_{\xi\xi} - B(h)\xi v_\xi = r(t)v\left(1-\frac{v}{K}\right), & (\xi,t)\in Q_T, \\ v_\xi(0,t) = v(1,t) = 0, & t\in(0,T), \\ v(\xi,0) = v_0(\xi), & \xi \in \overline{\Omega}, \\ h(t) = -\mu v_\xi(1,t), & t\in(0,T), \end{cases} \tag{2.5}$$

with

$$\begin{cases} A(h) = \frac{1}{2\int_0^t h(\tau)d\tau + s_0^2}, \\ B(h) = \frac{h}{2\int_0^t h(\tau)d\tau + s_0^2}. \end{cases} \tag{2.6}$$

We define

$$X_T = C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T) \times C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T], \tag{2.7}$$

and

$$||(v,h)||_{X_T} = ||v||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)} + ||h||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]}. \tag{2.8}$$

Theorem 2.1. Let $v_0 \in C^{2+\alpha}(\overline{\Omega})$, $v_0'(1) < 0$, $r \in C^{\frac{\alpha}{2}}[0,T]$. Then there exists a sufficient small $T_0 > 0$ such that the direct problem (2.5) has a unique solution $(v,h) \in X_T$ for any $0 < T < T_0$. Furthermore, we have the following estimate

$$||(v,h)||_{X_T} \le C\left[(T+T^2)\|r\|_{C^{\frac{\alpha}{2}}[0,T]} + ||v_0||_{C^{2+\alpha}(\overline{\Omega})}\right], \tag{2.9}$$

Where C is a constant depending on Ω, T, μ and s_0.

Proof. Define $D_{M,T} = V_{M,T} \times H_{M,T}$, where

$$V_{M,T} = \left\{\hat{v} \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T) \mid \hat{v}(\xi,0) = v_0(\xi)\,, ||\hat{v}||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)} \le M\right\},$$

$$H_{M,T} = \left\{\hat{h} \in C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T] \mid \hat{h}(t) = h^*\,, ||\hat{h}||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]} \le M\right\}, \tag{2.10}$$