

Hermite Wavelet based Galerkin Method for the Numerical Solutions of One Dimensional Elliptic Problems

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Abstract Wavelet analysis is a recently developed mathematical tool in applied mathematics. In this paper, we proposed the wavelet based Galerkin method for the numerical solution of one dimensional elliptic problems using Hermite wavelets. Here, Galerkin bases are constructed Hermite functions which are orthonormal bases and these are assumed bases elements which allow us to obtain the numerical solutions of the elliptic problems. The obtained numerical solutions are compared with the existing numerical methods and exact solution. Some of the test problems are considered to demonstrate the applicability and validity of the purposed method.

Keywords: Hermite wavelets; Galerkin method; Elliptic problems; Numerical solution.

1. Introduction

One dimensional elliptic problem occurs frequently in the fields of engineering and science. In most cases, we do not always find the exact solutions for these equations via analytical methods. In this case, it is very meaningful to give the high precision numerical solutions for this kind of problem by numerical methods. Recently, some of the numerical methods are used for the numerical solutions of boundary value problems. For example, Haar wavelet method [1], Legendre wavelet collocation method [2], Wavelet-Galerkin method [3], Laguerre wavelet Galerkin method [4] etc.

Wavelets theory is a new and has been emerging tool in applied mathematical research area. Its applications have been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets have generated significant interest from both theoretical and applied researchers over the last few decades. The concepts for understanding wavelets were provided by Meyer, Mallat, Daubechies, and many others, [5]. Since then, the number of applications where wavelets have been used has exploded. In areas such as approximation theory and numerical solutions of differential equations, wavelets are recognized as powerful weapons not just tools.

In general it is not always possible to obtain exact solution of an arbitrary differential equation. This necessitates either discretization of differential equations leading to numerical solutions, or their qualitative study which is concerned with deduction of important properties of the solutions without actually solving them. The Galerkin method is one of the best known methods for finding numerical solutions of differential equations and is considered the most widely used in applied mathematics [6]. Its simplicity makes it perfect for many applications. The wavelet-Galerkin method is an improvement over the standard Galerkin methods. The advantage of wavelet-Galerkin method over finite difference or finite element method has lead to tremendous applications in science and engineering. An approach to study differential equations is the use of wavelet function bases in place of other conventional piecewise polynomial trial functions in finite element type methods.

In this paper, we developed Hermite wavelet-Galerkin method (HWGM) for the numerical solution of differential equations and are based on expanding the solution by Hermite wavelets with unknown coefficients. The properties of Hermite wavelets together with the Galerkin method are utilized to evaluate the unknown coefficients and then a numerical solution of the differential equation is obtained.

The organization of the paper is as follows. In section 2, Preliminaries of Hermite wavelets are given. Hermite wavelet-Galerkin method of solution for the elliptic problem is given in section 3. In section 4 Numerical results are presented. Finally, conclusions of the proposed work are discussed in section 5.

2. Preliminaries of Hermite wavelets

Wavelets constitute a family of functions constructed from dialation and translation of a single function $\psi(x)$ called mother wavelet. When the dialation parameter a and translation parameter b varies continuously, we have the following family of continuous wavelets [7, 8]:

$$\psi_{a,b}(x) = |a|^{-1/2} \psi\left(\frac{x-b}{a}\right), \quad \forall \quad a, b \in \mathbb{R}, \quad a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 1$, $b_0 > 0$. We have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a|^{1/2} \psi(a^k x - nb_0), \quad \forall \quad a, b \in \mathbb{R}, \quad a \neq 0,$$

where $\psi_{k,n}$ form a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(x)$ forms an orthonormal basis. Hermite wavelets are defined as

$$\psi_{n,m}(x) = \begin{cases} \frac{2^{\frac{k}{2}}}{\sqrt{\pi}} \tilde{H}_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

$$\tilde{H}_m = \sqrt{\frac{2}{\pi}} H_m(x) \quad (2)$$

where $m = 0, 1, \dots, M-1$. In eq. (2) the coefficients are used for orthonormality. Here $H_m(x)$ are the second Hermite polynomials of degree m with respect to weight function $W(x) = \sqrt{1-x^2}$ on the real line $\mathbb{R}$ and satisfies the following recurrence formula $H_0(x) = 1$, $H_1(x) = 2x$,

$$H_{m+2}(x) = 2xH_{m+1}(x) - 2(m+1)H_m(x), \quad \text{where } m = 0, 1, 2, \dots \quad (3)$$

For $k = 1$ & $n = 1$ in (1) and (2), then the Hermite wavelets are given by

$$\begin{aligned} \psi_{1,0}(x) &= \frac{2}{\sqrt{\pi}}, \\ \psi_{1,1}(x) &= \frac{2}{\sqrt{\pi}}(4x - 2), \\ \psi_{1,2}(x) &= \frac{2}{\sqrt{\pi}}(16x^2 - 16x + 2), \\ \psi_{1,3}(x) &= \frac{2}{\sqrt{\pi}}(64x^3 - 96x^2 + 36x - 2), \\ \psi_{1,4}(x) &= \frac{2}{\sqrt{\pi}}(256x^4 - 512x^3 + 320x^2 - 64x + 2) \end{aligned}$$

And so on.

2.1 Function approximation

We would like to bring a solution function $u(x)$ under Hermite space by approximating $u(x)$ by elements of Hermite wavelet bases as follows,

$$u(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x) \quad (4)$$

where $\psi_{n,m}(x)$ is given in eq. (1).

We approximate $u(x)$ by truncating the series represented in Eq. (4) as,