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Existence of periodic solutions for second order delay differential equations with a singularity of repulsive type

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Abstract: In this paper, the problem of existence of periodic solution is studied for the second order delay differential equation with a singularity of repulsive type

$$x''(t) + f(x(t))x'(t) + \varphi(t)x(t-\tau_1) - g(x(t-\tau_2)) = h(t),$$

where τ_1 and τ_2 are constants, $g(x)$ is singular at $x = 0$, φ and h are $T-$periodic functions. By using a continuation theorem of coincidence degree theory, a new result on the existence of positive periodic solutions is obtained. The interesting is that the sign of function$\varphi(t)$ is allowed to change for$t \in [0, T]$.

Keywords: Liénard equation; Continuation theorem; Singularity; Periodic solution.

1. Introduction

The aim of this paper is to search for positive $T-$periodic solutions for second order delay differential equation with a singularity in the following form

$$x''(t) + f(x(t))x'(t) + \varphi(t)x(t-\tau_1) - g(x(t-\tau_2)) = h(t), \tag{1.1}$$

where τ_1 and τ_2 are constants, $f: [0, \infty) \to R$ is an arbitrary continuous function, $g \in C((0, +\infty), (0, +\infty))$ and $g(x)$ is singular of repulsive type at $x = 0$, i.e.,$g(x) \to +\infty$, as$x \to 0^+$, $\varphi, h: R \to R$ are $T-$periodic with function$h \in L^1([0, T], R), \varphi \in C([0, T], R)$, while the sign of function φ being changeable for $t \in [0, T]$.

In recent years, the problem of periodic solutions to the second order singular equation

$$x''(t) + f(x(t))x'(t) + \varphi(t)x(t-\tau_1) - \frac{b(t)}{x^{\lambda}(t)} = h(t), \tag{1.2}$$

where $f: [0, +\infty) \to R$is an arbitrary continuous function,$\varphi, b, h \in L^1[0, T]$ and $\lambda > 0$, has been studied widely. This is due to the fact that the singular term possesses a significant role in many practical situations [1-11]. For example, the singular term in the equations models the restoring force caused by a compressed perfect gas (see [3-6] and the references therein). Lazer and Solimini in the pioneering paper[12] first used the method of topological degree theory, together with the technique of upper and lower solutions, to study the existence of periodic solution to Eq.(1.2) where $f(x) \equiv 0, \varphi(t) \equiv 0, b(t) \equiv 1$. They obtained that if $\lambda \geq 1$,a necessary and sufficient condition for existence of a positive periodic solution to Eq.(1.2) is that$\bar{h} := \frac{1}{T}\int_0^T h(s)ds < 0$. After that, the problem of periodic solutions for singular differential equations like Eq.(1.2) has attracted the attention of many researchers[13-19]. We notice that the condition of $\varphi(t) \geq 0$ for a.e. $t \in [0, T]$ is required in [16-19], since it is crucial for obtain the priori estimates over all the possible periodic solutions to the equations

$$x''(t) + \lambda f(x(t))x'(t) + \lambda\varphi(t)x(t-\tau_1) - \frac{\lambda b(t)}{x^{\lambda}(t)} = \lambda h(t), \lambda \in (0,1). \tag{1.3}$$

We only find [20,21] where the sign of $\varphi(t)$ is allowed to change. In [20,21], a priori bounds of all the possible periodic solutions to Eq.(1.3) are estimated by using the inequality

$$\int_0^T \frac{u''(t)}{u^{\delta}(t)} dt \geq 0, \tag{1.4}$$

where $\delta > 0$is an arbitrary constant, $u(t)$ is a positive $T-$periodic function with $u \in C^2([0, T], R)$.

Motivated by this, in this paper, we study the existence of positive $T-$periodic solution for the equation (1.1). Since there is a delay τ_1 in (1.1), generally, the inequality like (1.4) for

$$\delta = 1$$

$$\int_0^T \frac{u''(t)}{u(t-\tau_1)} dt \geq 0.$$

may not hold. This means that the work to estimate a priori bounds of all the possible periodic solutions to the equations

$$x''(t)+\lambda f(x(t))x'(t)+\lambda\varphi(t)x(t-\tau_1)-\lambda g(x(t-\tau_2))=\lambda h(t),\lambda\in(0,1).$$

is more difficult than the corresponding ones associated to (1.3).

2. Preliminary lemmas

Throughout this paper, let $C_T=\{x\in C(R,R): x(t+T)=x(t) \text{ for all } t\in R\}$ with the norm defined by $|x|_\infty=\max_{t\in[0,T]}|x(t)|$. For any T −periodic solution $y(t)$ with $y\in L^1([0,T],R)$, $y_+(t)$ and $y_-(t)$ is denoted by $max\{y(t),0\}$ and $-min\{y(t),0\}$ respectively, and $\overline{y}=\frac{1}{T}\int_0^T y(s)ds$. Clearly, $y(t)=y_+(t)-y_-(t)$ for all $t\in R$, and $\overline{y}=\overline{y_+}-\overline{y_-}$.

The following Lemma is the consequence of Theorem 3.1 in [22].

Lemma 2.1. Assume that there exist positive constants M_0, M_1 and M_2 with $0<M_0<M_1$, such that the following conditions hold.

1.For each $\lambda\in(0,1]$, each possible positive T −periodic solution x to the equation

$$u''(t)+\lambda f(u(t))u'(t)+\lambda\varphi(t)u(t-\tau_1)-\lambda g(u(t-\tau_2))=\lambda h(t),$$

satisfies the inequalities $M_0<x(t)<M_1$ and $|x'(t)|<M_2$ for all $t\in[0,T]$.

2.Each possible solution c to the equation

$$g(c)-c\overline{\varphi}+\overline{h}=0,$$

satisfies the inequality $M_0<c<M_1$.

3.It holds

$$(g(M_0)-\overline{\varphi}M_0+\overline{h})(g(M_1)-\overline{\varphi}M_1+\overline{h})<0,$$

Then Eq.(1.1) has at least one T −periodic solution u such that $M_0<u(t)<M_1$ for all $t\in[0,T]$.

Lemma 2. 2.[19] Let x be a continuous T −periodic continuous differential function. Then, for any $\tau\in(0,T]$,

$$\left(\int_0^T|x(s)|^2ds\right)^{\frac{1}{2}}\le\frac{T}{\pi}\left(\int_0^T|x'(s)|^2ds\right)^{\frac{1}{2}}+\sqrt{T}|x(\tau)|.$$

In order to study the existence of positive periodic solutions to Eq.(1.1), we list the following assumptions.

$[H_1]$ The function $\varphi(t)$ satisfies the following conditions

$$\int_0^T\varphi_+(s)ds>0,\sigma:=\frac{\int_0^T\varphi_-(s)ds}{\int_0^T\varphi_+(s)ds}\in[0,1) \text{ and } \sigma_1:\frac{T^{\frac{1}{2}}}{1-\sigma}\left(\int_0^T\varphi_+(t)dt\right)^{\frac{1}{2}}\in(0,1);$$

$[H_2]$ there are constants $M>0$ and $A>0$ such that $g(x)\in(0,A)$ for all $x>M$;

$[H_3]$ $\int_0^1 g(s)ds=+\infty$;

$[H_4]$ $\lim_{x\to0^+}g(x)=+\infty$.

Remark 2.1. It is noted that assumption $[H_4]$ can not be deduced from assumption $[H_3]$. For example, let $g(x)\frac{1}{x}|sin\frac{1}{x}|$ for all $x\in(0,+\infty)$, then assumption $[H_3]$ is satisfied. But, assumption $[H_4]$ does not hold.

Remark 2.2. If assumptions $[H_1]$-$[H_2]$ and $[H_4]$ hold, then there are constants D_1 and D_2 with $0<D_1<D_2$ such that

$$g(x)-\overline{\varphi}x+\overline{h}>0 \text{ for all } x\in(0,D_1)$$

and

$$g(x)-\overline{\varphi}x+\overline{h}<0 \text{ for all } x\in(D_2,+\infty)$$

Now, we suppose that assumptions $[H_1]$ and $[H_2]$ hold, and embed Eq.(1.1) into the following equations family with a parameter $\lambda\in(0,1)$

$$x''(t)+\lambda f(x(t))x'(t)+\lambda\varphi(t)x(t-\tau_1)-\lambda g(x(t-\tau_2))=\lambda h(t),\lambda\in(0,1]. \tag{2.1}$$

Let

$$\Omega=\{x\in C_T: x''(t)+\lambda f(x(t))x'(t)+\lambda\varphi(t)x(t-\tau_1)-\lambda g(x(t-\tau_2))=\lambda h(t),\lambda\in(0,1], \\ x(t)>0,\forall t\in[0,T]\},$$

and