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Error Analysis for Sparse Time-Frequency Decomposition of Non-Integer Period Sampling Signals

Qi Yang [1+]

[1] *Department of Mathematics, Jinan University, Guangzhou, 510632, China*
+ E-mail: yangqi@stu2016.jnu.edu.cn
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Abstract. In this paper, we review a nonlinear matching pursuit approach (Hou and Shi, 2013), a data-driven time-frequency analysis method, which is looking for the sparsest representation of multiscale data over a dictionary consisting of all intrinsic mode functions (IMFs). In many practical problems, signals are non-integer period sampled. In other words, the time window may not contain exactly an integer number of signal periods. We consider the sparse time-frequency decomposition of non-integer period sampling signals by the nonlinear matching pursuit method and estimate the error. The estimation show that the relative error depends on the separation factor, the frequency ratio, and the number of periods of the IMF.

Keywords: sparse time-frequency decomposition, non-integer period sampling, scale separation

1. Introduction

Data is ubiquitous in our lives, and how to extract useful information from multiscale data has aroused research upsurge in recent years. The Fourier transform is a conventional method for analyzing linear and stationary data by decompose the signal into a linear combination of different frequency components. While the Fourier transform is widely applied in many fields, but it is still has some limitations in processing oscillatory data. When traditional data analysis methods can not provide simple indices with clear physical meanings, new ideas are in need. Time-frequency analysis method represents one-dimensional time signal into time-frequency (TF) domains, which is an important tool for nonstationary and nonlinear signal analysis. These include the short time Fourier transform, the wavalet transform [17], and the Wigner-Ville distribution [15]. The analytic signal (AS) method is another important approach in TF analysis was introduced by Van der Pol [2] and Gabor [7], and it has shown its usefulness in many applications, see, e.g. [11, 16, 1]. The AS method contains the original function and its Hilbert transform. In addition, an analytic signal has only the non-negative frequency components which has more physical meaning. In the past decades, the empirical mode decomposition (EMD) [13] was proposed to study the dynamics hidden inside an oscillatory signal which has offered an effective method for nonstationary data analysis. The EMD algorithm decomposes a signal into several intrinsic mode functions (IMFs) via a sifting process. The EMD method has shown its usefulness in analysing signal components caused by various sources, see, e.g., [5, 14, 26, 27]. But its mathematical foundation is still lacking cannot be ignored, which is due to the empirical nature. Recently, many variations of EMD were proposed, like the ensemble empirical mode decomposition (EEMD) [25], the variational mode decomposition [12], the synchrosqueezed wavelet transforms [10], etc.

Inspired by the EMD method and compressive sensing [6, 8, 9], Hou and Shi proposed a data-driven time-frequency analysis method [21, 22]. The data-driven time-frequency analysis method based on the sparsest representation of multiscale data to decompose the signal into a finite number of IMF with a small residual:

$$f(t)=\sum_{k=1}^{M} a_k(t)\cos\theta_k(t)+r(t), \quad t\in[0,T], \tag{1.1}$$

where each IMF is a signal with amplitude modulation-frequency modulation (AM-FM). The amplitude $a_k(t)>0$, the instantaneous frequency $\omega_k(t)=\theta_k'(t)>0$, and $r(t)$ is a small residual.

The idea of this method is looking for the sparsest representation over the dictionary consisting of all IMFs by solving the following nonlinear optimization problem:

$$\min_{(a_k)_{1\le k\le M},\,(\theta_k)_{1\le k\le M}} M \quad \text{subject to} \quad f(t)=\sum_{k=1}^{M} a_k(t)\cos\theta_k(t),\ 0\le t\le T,\ a_k\cos\theta_k\in D, \tag{P$_0$}$$

where D is a dictionary which is defined as

$$D=\{a(t)\cos\theta(t):\theta'(t)\ge 0,\ a(t)\in V(\theta)\}, \tag{1.2}$$

$V(\theta)$ is a is the collection of all the function that are less oscillatory than $\cos\theta(t)$. Liu et al analyzed the uniqueness of the optimization problem (P_0) under the assumption of the scale separation by considering a simplified slow evolution chirp model and studying the wavelet transform of each IMF; it is proved that under the assumption of the scale separation and the signal $f(t)$ is well separated, the solution of (P_0) is unique up to an error determined by the scale separation property [3]. Now, we review the definitions of scale separation and well-separated signal.

Definition 1.1 (scale separation [3]). A function $f(t)=a(t)\cos\theta(t)\ (t\in[0,T])$ is said to satisfy a scale separation property with a separation factor $\varepsilon>0$, if $a(t)$ and $\theta(t)$ satisfy the following conditions:

$$\begin{aligned} &a(t)\in C^1(\mathbb{R}),\ \theta\in C^2(\mathbb{R}),\ \inf_{t\in[0,T]}\theta'(t)>0,\\ &\frac{\sup_{t\in[0,T]}\theta'(t)}{\inf_{t\in[0,T]}\theta'(t)}\le M'<+\infty,\ \left|\frac{a'(t)}{a(t)\theta'(t)}\right|\le\varepsilon,\ \left|\frac{\theta''(t)}{(\theta'(t))^2}\right|\le\varepsilon,\quad \forall t\in[0,T]. \end{aligned} \tag{1.3}$$

In the above definition, the condition $\inf\theta'(t)>0(t\in[0,T])$ is to ensure that the instantaneous frequency has a physical meaning and the condition $|a'(t)/(a(t)\theta'(t))|\le\varepsilon$ implies the envelope $a(t)$ is smoother than the phase function $\theta(t)$. The variation of the instantaneous frequency $\theta'(t)$ over the time domain is bounded; otherwise, it is more likely to cause the mode mixing. The definition of well separated signal as follows.

Definition 1.2 (well-separated signal [3]). A signal $f:[0,T]\to\mathbb{R}$ is said to be well separated with separation factor ε and frequency ratio β if it can be written as

$$f(t)=\sum_{k=1}^{M} a_k(t)\cos\theta_k(t)+r(t),$$

where all $f_k(t)=a_k(t)\cos\theta_k(t)$ satisfy the scale separation property with separation factor ε, $r(t)=O(\varepsilon)$, and their phase function $\theta_k(t)$ satisfy

$$\theta_k'(t)\ge\beta\theta_{k-1}'(t),\quad \forall t\in[0,T],$$

and $\beta>1,\beta-1=O(1)$.

The optimization problem (P_0) can be seen as a nonlinear version of the L^0 minimization problem which is NP-hard and challenging to solve [21, 22, 23]. Mallat and Zhang introduced an algorithm, called matching pursuit, that builds up a sequence of sparse approximations stepwise and provide the representation of (1.1) [8]. For Gabor dictionaries, S. Qian and D. Chen proposed a similar algorithm [19]. The basis pursuit (BP) is an important optimization principle for decomposing a signal, which was introduced by S. Chen, D. Donoho and M. Saunders [20]. Because the L^1 norm is approximately equivalent to the L^0 norm under certain conditions, the main idea of BP is to find a representation of the signal whose coefficients have the smallest L^1 norm.

It is too difficult to solve the optimization problem (P_0) which is nonlinear and nonconvex. It is like the EMD method, Hou and Shi introduced an algorithm based on matching pursuit to solve problem (P_0) [23, 24]. This nonlinear matching pursuit (NMP) algorithm is stated as follows.