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Numerical Computations of Eleventh Order Boundary Value Problems with Bezier Polynomials by Galerkin Weighted Residual Method

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Abstract: Some techniques are available to solve numerically higher order boundary value problems. The aim of this paper is to apply Galerkin weighted residual method (GWRM) for solving eleventh order linear and nonlinear boundary value problems. Using GWRM, approximate solutions of eleventh-order boundary value problems are developed. This approach provides the solution in terms of a convergent series. Approximate results are given for several examples to illustrate the implementation and accuracy of the method. The results are depicted both graphically and numerically. All results are compared with the analytical solutions to show the convergence of the proposed algorithm. It is observed that the present method is a more effective tool and yields better results. All problems are computed using the software MATLAB R2017a.

Keywords: Differential Equations; Numerical solutions; Galerkin method; Bezier polynomials.

1. Introduction

Higher order boundary value problems (BVPs) occur in the study of fluid dynamics, astrophysics, hydrodynamic, hydro magnetic stability, astronomy, beam and long wave theory, induction motors, engineering, and applied physics. The boundary value problems of higher order have been examined due to their mathematical importance and applications in diversified applied sciences [1-2]. Twizell et al [3] developed numerical methods for eight, tenth and twelfth order eigenvalue problems arising in thermal instability. Scott and Watts [4] developed a numerical method for the solution of linear BVPs using a combination of superposition and orthonormalization. Siddiqi et al [5] used Variational iteration technique to obtain numerical approximations for eleventh-order BVPs by converting the original problem into a system of integral equations. Very recently Amjad Hussain et al [6] derived the numerical solutions of eleventh-order BVPs using differential transformation method. Siddiqi and Ghazala [7-10] presented the solutions of eight, tenth and twelfth order boundary value problems using spline and Non-polynomial spline.

In the present paper, the eleventh order boundary value problems are solved using the Galerkin weighted residual method. The problem has the following form:

$$c_{11}\frac{d^{11}u}{dx^{11}}+c_{10}\frac{d^{10}u}{dx^{10}}+c_9\frac{d^9u}{dx^9}+c_8\frac{d^8u}{dx^8}+c_7\frac{d^7u}{dx^7}+c_6\frac{d^6u}{dx^6}+c_5\frac{d^5u}{dx^5}+c_4\frac{d^4u}{dx^4}+c_3\frac{d^3u}{dx^3}+c_2\frac{d^2u}{dx^2}+c_1\frac{du}{dx}+c_0u=r,\ a<x<b \quad (1a)$$

subject to the following boundary conditions:

$$u(\mathrm{a})=\mathrm{A}_0,\ u(b)=\mathrm{B}_0,\ u'(a)=\mathrm{A}_1,\ u'(b)=\mathrm{B}_1,\ u''(a)=\mathrm{A}_2,\ u''(b)=\mathrm{B}_2,\ u'''(a)=\mathrm{A}_3,$$
$$u'''(b)=\mathrm{B}_3,\ u^{(iv)}(a)=\mathrm{A}_4,\ u^{(iv)}(b)=\mathrm{B}_4,\ u^{(v)}(a)=\mathrm{A}_5 \quad (1b)$$

Where $\mathrm{A}_i, i=0,1,2,3,4,5$ and $\mathrm{B}_j, j=0,1,2,3,4$ are finite real constants and $\mathrm{c}_i, i=0,1,\ldots,11$ and r are all continuous and differentiable functions of x defined on the interval [a, b].

The paper is organized in four sections. In section 2, we give a short description on Bezier polynomials. The analysis of Galerkin weighted residual method is discussed in section 3. In section 4, three numerical examples are presented to assess the efficiency of the Galerkin weighted residual technique.

2. Bezier Polynomials

The general form of the Bezier polynomials of nth degree over the interval [0, 1] is defined by

$$B_{j,n}(x)=\sum_{j=0}^{n}\binom{n}{j}x^j(1-x)^{n-j}P_j\,, 0\le x\le 1$$

Where the binomial coefficients are given by

$$\binom{n}{j} = \frac{n!}{(n-j)!j!}$$

The points P_j are called control points for the Bezier curve.

We write first few Bezier polynomials over the interval [0,1]:

$B_0(x) = (1-x)^{19}$, $B_1(x) = 19(1-x)^{18}x$, $B_2(x) = 171(1-x)^{17}x^2$, $B_3(x) = 969(1-x)^{16}x^3$
$B_4(x) = 3876(1-x)^{15}x^4$, $B_5(x) = 11628(1-x)^{14}x^5$, $B_6(x) = 27132(1-x)^{13}x^6$, $B_7(x) = 50388(1-x)^{12}x^7$
$B_8(x) = 75582(1-x)^{11}x^8$, $B_9(x) = 92378(1-x)^{10}x^9$, $B_{10}(x) = 92378(1-x)^9x^{10}$, $B_{11}(x) = 3876(1-x)^8x^{11}$
$B_{12}(x) = 75582(1-x)^7x^{12}$, $B_{13}(x) = 27132(1-x)^6x^{13}$, $B_{14}(x) = 11628(1-x)^5x^{14}$, $B_{15}(x) = 3876(1-x)^4x^{15}$
$B_{16}(x) = 969(1-x)^3x^{16}$, $B_{17}(x) = 171(1-x)^2x^{17}$, $B_{18}(x) = 19(1-x)x^{18}$, $B_{19}(x) = x^{19}$

Note that each of these $n+1$ polynomials having degree n satisfies the following properties:

(i) $B_{j,n}(x) = 0$ if $j < 0$ or $j > n$

(ii) $\sum_{j=0}^{n} B_{j,n}(x) = 1$

(iii) $B_{j,n}(a) = B_{j,n}(b) = 0,\ j = 1,2,\ldots,n-1$

For these properties, Bezier polynomials are used in the trail functions satisfying the corresponding homogeneous form of the essential boundary conditions in the Galerkin weighted residual method to solve a BVP.

3. Matrix Formulation of Eleventh-order BVPs

In this section, we first derived the matrix formulation for eleventh-order linear BVP and then we extend our idea for solving nonlinear BVP. To solve the boundary value problem (1) by the Galerkin weighted residual method we approximate $\tilde{u}(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} \beta_i B_i(x),\ n \geq 2 \qquad (2)$$

Here $\theta_0(x)$ is specified by the essential boundary conditions and $B_i(\mathrm{a}) = B_i(b) = 0$, for each $i = 1,2,3,\ldots,n-1$.

Using (2) into (1), the Galerkin weighted residual equations are:

$$\int_{\mathrm{a}}^{b}\left[c_{11}\frac{d^{11}\tilde{u}}{dx^{11}} + c_{10}\frac{d^{10}\tilde{u}}{dx^{10}} + c_9\frac{d^9\tilde{u}}{dx^9} + c_8\frac{d^8\tilde{u}}{dx^8} + c_7\frac{d^7\tilde{u}}{dx^7} + c_6\frac{d^6\tilde{u}}{dx^6} + c_5\frac{d^5\tilde{u}}{dx^5} + c_4\frac{d^4\tilde{u}}{dx^4} + c_3\frac{d^3\tilde{u}}{dx^3} + c_2\frac{d^2\tilde{u}}{dx^2} + c_1\frac{d\tilde{u}}{dx} + c_0\tilde{u} - r\right]B_j(x)dx = 0,\ j = 1,2,..,n-1 \qquad (3)$$

Integrating by parts the terms up to second derivative on the left hand side of (3), we get

$$\int_{\mathrm{a}}^{b} c_{11}\frac{d^{11}\tilde{u}}{dx^{11}}B_j(x)dx = -\left[\frac{d}{dx}[c_{11}B_j(x)]\frac{d^9\tilde{u}}{dx^9}\right]_{\mathrm{a}}^{b} + \left[\frac{d^2}{dx^2}[c_{11}B_j(x)]\frac{d^8\tilde{u}}{dx^8}\right]_{\mathrm{a}}^{b} - \left[\frac{d^3}{dx^3}[c_{11}B_j(x)]\frac{d^7\tilde{u}}{dx^7}\right]_{\mathrm{a}}^{b}$$
$$+ \left[\frac{d^4}{dx^4}[c_{11}B_j(x)]\frac{d^6\tilde{u}}{dx^6}\right]_{\mathrm{a}}^{b} - \left[\frac{d^5}{dx^5}[c_{11}B_j(x)]\frac{d^5\tilde{u}}{dx^4}\right]_{\mathrm{a}}^{b} + \left[\frac{d^6}{dx^6}[c_{11}B_j(x)]\frac{d^4\tilde{u}}{dx^4}\right]_{\mathrm{a}}^{b} - \left[\frac{d^7}{dx^7}[c_{11}B_j(x)]\frac{d^3\tilde{u}}{dx^3}\right]_{\mathrm{a}}^{b}$$
$$+ \left[\frac{d^8}{dx^8}[c_{11}B_j(x)]\frac{d^2\tilde{u}}{dx^2}\right]_{\mathrm{a}}^{b} - \left[\frac{d^9}{dx^9}[c_{11}B_j(x)]\frac{d\tilde{u}}{dx}\right]_{\mathrm{a}}^{b} + \int_{\mathrm{a}}^{b}\frac{d^{10}}{dx^{10}}[c_{11}B_j(x)]\frac{d\tilde{u}}{dx}dx \qquad (4)$$

$$\int_{\mathrm{a}}^{b} c_{10}\frac{d^{10}\tilde{u}}{dx^{10}}B_j(x)dx = -\left[\frac{d}{dx}[c_{10}B_j(x)]\frac{d^8\tilde{u}}{dx^8}\right]_{\mathrm{a}}^{b} + \left[\frac{d^2}{dx^2}[c_{10}B_j(x)]\frac{d^7\tilde{u}}{dx^7}\right]_{\mathrm{a}}^{b} - \left[\frac{d^3}{dx^3}[c_{10}B_j(x)]\frac{d^6\tilde{u}}{dx^6}\right]_{\mathrm{a}}^{b}$$
$$+ \left[\frac{d^4}{dx^4}[c_{10}B_j(x)]\frac{d^5\tilde{u}}{dx^5}\right]_{\mathrm{a}}^{b} - \left[\frac{d^5}{dx^5}[c_{10}B_j(x)]\frac{d^4\tilde{u}}{dx^4}\right]_{\mathrm{a}}^{b} + \left[\frac{d^6}{dx^6}[c_{10}B_j(x)]\frac{d^3\tilde{u}}{dx^3}\right]_{\mathrm{a}}^{b} - \left[\frac{d^7}{dx^7}[c_{10}B_j(x)]\frac{d^2\tilde{u}}{dx^2}\right]_{\mathrm{a}}^{b}$$
$$+ \left[\frac{d^8}{dx^8}[c_{10}B_j(x)]\frac{d\tilde{u}}{dx}\right]_{\mathrm{a}}^{b} - \int_{\mathrm{a}}^{b}\frac{d^9}{dx^9}[c_{10}B_j(x)]\frac{d\tilde{u}}{dx}dx \qquad (5)$$

$$\int_{\mathrm{a}}^{b} c_9\frac{d^9\tilde{u}}{dx^9}B_j(x)dx = -\left[\frac{d}{dx}[c_9B_j(x)]\frac{d^7\tilde{u}}{dx^7}\right]_{\mathrm{a}}^{b} + \left[\frac{d^2}{dx^2}[c_9B_j(x)]\frac{d^6\tilde{u}}{dx^6}\right]_{\mathrm{a}}^{b} - \left[\frac{d^3}{dx^3}[c_9B_j(x)]\frac{d^5\tilde{u}}{dx^5}\right]_{\mathrm{a}}^{b}$$
$$+ \left[\frac{d^4}{dx^4}[c_9B_j(x)]\frac{d^4\tilde{u}}{dx^4}\right]_{\mathrm{a}}^{b} - \left[\frac{d^5}{dx^5}[c_9B_j(x)]\frac{d^3\tilde{u}}{dx^3}\right]_{\mathrm{a}}^{b} + \left[\frac{d^6}{dx^6}[c_9B_j(x)]\frac{d^2\tilde{u}}{dx^2}\right]_{\mathrm{a}}^{b} - \left[\frac{d^7}{dx^7}[c_9B_j(x)]\frac{d\tilde{u}}{dx}\right]_{\mathrm{a}}^{b}$$
$$+ \int_{\mathrm{a}}^{b}\frac{d^8}{dx^8}[c_9B_j(x)]\frac{d\tilde{u}}{dx}dx \qquad (6)$$

$$\int_{\mathrm{a}}^{b} c_8\frac{d^8\tilde{u}}{dx^8}B_j(x)dx = -\left[\frac{d}{dx}[c_8B_j(x)]\frac{d^6\tilde{u}}{dx^6}\right]_{\mathrm{a}}^{b} + \left[\frac{d^2}{dx^2}[c_8B_j(x)]\frac{d^5\tilde{u}}{dx^5}\right]_{\mathrm{a}}^{b} - \left[\frac{d^3}{dx^3}[c_8B_j(x)]\frac{d^4\tilde{u}}{dx^4}\right]_{\mathrm{a}}^{b}$$
$$+ \left[\frac{d^4}{dx^4}[c_8B_j(x)]\frac{d^3\tilde{u}}{dx^3}\right]_{\mathrm{a}}^{b} - \left[\frac{d^5}{dx^5}[c_8B_j(x)]\frac{d^2\tilde{u}}{dx^2}\right]_{\mathrm{a}}^{b} + \left[\frac{d^6}{dx^6}[c_8B_j(x)]\frac{d\tilde{u}}{dx}\right]_{\mathrm{a}}^{b} - \int_{\mathrm{a}}^{b}\frac{d^7}{dx^7}[c_8B_j(x)]\frac{d\tilde{u}}{dx}dx \qquad (7)$$

$$\int_{\mathrm{a}}^{b} c_7\frac{d^7\tilde{u}}{dx^7}B_j(x)dx = -\left[\frac{d}{dx}[c_7B_j(x)]\frac{d^5\tilde{u}}{dx^5}\right]_{\mathrm{a}}^{b} + \left[\frac{d^2}{dx^2}[c_7B_j(x)]\frac{d^4\tilde{u}}{dx^4}\right]_{\mathrm{a}}^{b} - \left[\frac{d^3}{dx^3}[c_7B_j(x)]\frac{d^3\tilde{u}}{dx^3}\right]_{\mathrm{a}}^{b}$$
$$+ \left[\frac{d^4}{dx^4}[c_7B_j(x)]\frac{d^2\tilde{u}}{dx^2}\right]_{\mathrm{a}}^{b} - \left[\frac{d^5}{dx^5}[c_7B_j(x)]\frac{d\tilde{u}}{dx}\right]_{\mathrm{a}}^{b} + \int_{\mathrm{a}}^{b}\frac{d^6}{dx^6}[c_7B_j(x)]\frac{d\tilde{u}}{dx}dx \qquad (8)$$