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The nonexistence of global solutions for a damped time fractional diffusion equation with nonlinear memory

Qian Qian and Yaning Li +

School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

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Abstract: In this paper, we study the non-global existence of solutions to the following time fractional nonlinear diffusion equations

$$\begin{cases} {}^C D_{0|t}^{\alpha} u - \triangle u + (1+t)^r u_t = I_{0|t}^{\beta}(|u|^{p-1}u), \ x \in \mathbb{R}^n, \ t > 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \ x \in \mathbb{R}^n, \end{cases}$$

where $1 < \alpha < 2$, $\beta \in (0,1)$, $1 < \alpha + \beta < 2$, $r \in (-1,1)$, $p > 1$, $u_0, u_1 \in L^q(\mathbb{R}^n)(q > 1)$ and ${}^C D_{0|t}^{\alpha} u$ denotes left Caputo fractional derivative of order α . By using the test function method, we prove that the problem admits no global weak solution with suitable initial data when p falls in different intervals. Our results generalize that in [4].

Keywords: fractional derivative, blow-up, test function, nonlinear memory.

1. Introduction

Fractional differential equations are widely used to describe abnormal diffusion, Hamiltonian chaos, dynamical systems with chaotic mechanical behavior and so on, ect. see [5, 10, 12] and the references therein. In addition, the fractional evolution equations of time appearing in electromagnetic, acoustic and mechanical phenomena have also attracted much attention. Such equations replace the first time derivative with the fractional derivative of α , where α belongs to (0,1). In recent years, time fractional differential equations yield many different results, see [1, 7, 8, 13, 14, 15] and the references therein. For instance, in [11], the existence and properties of solutions of time fractional equations in bounded domains are considered by using the expansion of characterisitic functions. In [2], the quasilinear abstract time fractional development equation in continuous interpolation space is studied. In [13], the L^p -type maximum regularity results for abstract parabolic Voltera equations with inhomogeneous boundary data problems are established by using pure operator theory. In [6], the $L^p(L^q)$ theory of semilinear time fractional equations with variable coefficients is given by using the classical theory of partial differential equation theory, such as the Marcinkiewicz interpolation theorem, the Calderon-Zygmund theorem, and the perturbation arguments.

This paper is concerned with the non-global existence of solutions to the Cauchy problem for a nonlinear time-fractional with nonlinear memory

$$\begin{cases} {}^C D_{0|t}^{\alpha} u - \triangle u + (1+t)^r u_t = I_{0|t}^{1-\gamma}(|u|^{p-1}u), \ x \in \mathbb{R}^N, \ t > 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \ x \in \mathbb{R}^N, \end{cases} \tag{0. 1}$$

where $1 < \alpha < 2$, $\beta \in (0,1)$, $1 < \alpha + \beta < 2$, $r \in (-1,1)$, $p > 1$, $u_0, u_1 \in L^q(R^n)(q > 1)$ and ${}^C D_{0|t}^{\alpha} u$

+ Corresponding author. Tel:17705183752
E-mail address: liyn11@126.com.

denotes left Riemann-Liouville fractional derivative of order α, $I_{0|t}^{\beta}$ denotes left Riemann-Liouville fractional integrals of order β and is defined by

$$I_{0|t}^{\beta}u=\frac{1}{\Gamma(1-\beta)}\int_0^t(t-s)^{1-\beta}u(s)ds.$$

In the case $\alpha=2$, $\beta=1-\gamma$, the damped wave equation

$$\begin{cases}u_{tt}-\triangle u+(1+t)^r u_t=I_{0|t}^{1-\gamma}(|u|^{p-1}u),\ (t,x)\in\mathbb{R}^n,\\ u(0,x)=u_0(x),u_t(0,x)=u_1(x),\ \ x\in\mathbb{R}^n.\end{cases}\tag{0.2}$$

has been studied by [4], they proved the blow-up results for local (in time) Sobolev solutions. And applied the test function method to showed that when $\int_{R^n}u_0(x)dx>0$ and $\int_{R^n}u_1(x)dx>0$, if

$$\frac{p}{p-1}>\inf_{d>0}\max\{\frac{\frac{nd}{2}+1}{1-\gamma+d},\sqrt{\frac{\frac{nd}{2}+1}{(1-\gamma)(1-r)}+(\frac{1-\gamma-(1-r)\frac{nd}{2}}{2(1-\gamma)(1-r)})^2}-\frac{1-\gamma-(1-r)\frac{nd}{2}}{2(1-\gamma)(1-r)}\}$$

for $r\in(-1,0)$ or $r\in(0,1)$, or $\frac{p}{p-1}>\frac{\frac{n}{2}+1}{2-\gamma}$ for $r=0$, the weak solution of (1.2) do not exist global in time. In the results of our paper, when $\alpha=2$, that is what the [4] says.

Recently, there are many papers which considered the existence and nonexistence of the global solution to semilinear time fractional diffusion equation and diffusion equation with nonlinear memory.

In [3], using the test function method, Fino and Kirane considered a heat equation with nonlinear memory. They generalized test function method to fractional case and determined the Fujita critical exponent of the problem.

For the nonlinear time fractional diffusion equation (i.e. (1.1) with $\gamma=1$ and the damped term $(1+t)^r u_t$ do not exist),

$$\begin{cases}{}^{C}D_{0|t}^{\alpha}u-\triangle u=|u|^{p-1}u,\ \ x\in\mathbb{R}^N,\ \ t>0,\\ u(0,x)=u_0(x)\geq 0,\ \ x\in\mathbb{R}^N.\end{cases}\tag{0.3}$$

Zhang and Sun [16] studied the local existence of this problem, where $u_0\in C_0(R^N)$, they obtained that if $1<p<1+\frac{2}{N}$, u blows up in finite time, and if $p\geq 1+\frac{2}{N}$, the problem (1.3) exists a global solution for small initial data. It should be noted that in the critical case $p=1+\frac{2}{N}$, the solution of (1.3) can exist globally. In [15], Zhang and Li studied the local existence and uniqueness of mild solutions of problem (1.3), and used test function method to show the blow-up and global existence of the solutions to (1.3).

As far as we know, there are few paper consider the existence and non-existence for solutions of damped fractional diffusion equation. Motivated by the above results, in this paper, we study the non-global existence of problem (1.1). Particularly, for $u_0,u_1\in L^q(R^n)(q>1)$, $p+q=1$, $u_0>0$, $u_0\not\equiv 0$, $\chi(x)=(\int_{R^n}e^{-\sqrt{n^2+|x|^2}}dx)^{-1}e^{-\sqrt{n^2+|x|^2}}$, we will show that with the following two conditions,