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Analytic solutions of a class of matrix minimization model with unitary constraints

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Abstract: In this paper we present analytic solutions of a class of matrix minimization model with unitary constraints as follows:

$$\min_{U_k\in\mathbf{U}_n, W_k\in\mathbf{U}_t, V_k\in\mathbf{U}_m} |\det(cI_m \pm \prod_{k=1}^{s} A_k U_k B_k W_k C_k V)|$$

$$\min_{U_k\in\mathbf{U}_n, W_k\in\mathbf{U}_t, V_k\in\mathbf{U}_m} |\mathrm{tr}(cI_m \pm \prod_{k=1}^{s} A_k U_k B_k W_k C_k V)|,$$

where $A_k \in \mathbf{C}^{m\times n}$, $B_k \in \mathbf{C}^{n\times t}$, $C_k \in \mathbf{C}^{t\times m}$, $\mathbf{C}^{m\times n}$ denotes $m\times n$ complex matrix set, and c is a complex number, I_m denotes the m-order identity matrix, $\det(\cdot)$ and $\mathrm{tr}(\cdot)$ denote matrix determinant and trace function, respectively. The proposed results improve some existing ones in Xu (2019) [1]. Numerical examples are given to verify the validity of the theoretical results.

Keywords: constrained matrix minimization model, determinant function, trace function, unitary constraints.

1. Introduction

The matrix optimization model with unitary constraints has important applications in Kronecker canonical form of a general matrix pencil, linearly constrained least-squares problem, test signals of mechanical systems, and aero engine fault diagnosis, see [2,3,4,5].

The latest significant application of matrix optimization model with unitary constraints is in the data analysis of DNA micro-array analysis [6,7,8,9]. Xu [1] in 2019 considered the upper bound of chordal metric between generalized singular values of Grassman matrix pairs with the same number of columns, which can be applied in comparing two sets of DNA micro-arrays of different organisms. Motivated by the applications, in this paper we consider analytic solutions of a class of matrix minimization model with different dimensional unitary constraints. The considered matrix minimization model are as follows:

$$\min_{U_k\in\mathbf{U}_n, W_k\in\mathbf{U}_t, V_k\in\mathbf{U}_m} |\det(cI_m \pm \prod_{k=1}^{s} A_k U_k B_k W_k C_k V)|, \tag{1.1}$$

$$\min_{U_k\in\mathbf{U}_n, W_k\in\mathbf{U}_t, V_k\in\mathbf{U}_m} |\det(cI_m \pm \prod_{k=1}^{s} A_k U_k B_k W_k C_k V)|, \tag{1.2}$$

where c is a complex number and A_k, B_k, C_k are $m\times n, n\times t, t\times m$ complex matrices, respectively. In this paper we will discuss their analytic solutions.

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1.1. Literature review

The existing works related to the constrained matrix problems (1.1) and (1.2) are summarized as follows. John von Neumann in 1937 [10] and K. Fan in 1951 [11] studied the maximum value problem of trace function for the same dimensional matrix. They presented

$$\max_{U_1,\cdots,U_m\in\mathbf{U_n}} Re\left[\mathrm{tr}\prod_{j=1}^{m}U_jA_j\right]=\max_{U_1,\cdots,U_m\in\mathbf{U_n}}|\mathrm{tr}\prod_{j=1}^{m}U_jA_j|=\sum_{i=1}^{n}\prod_{j=1}^{m}\sigma_i(A_j), \tag{1.3}$$

where $Re[x]$ denotes the real part of the complex number x. A special case of (1.3) was studied by Lu [12], where $m=2$, and both A_1 and A_2 are positive diagonal matrices with the main diagonal elements between 0 and 1 descending simultaneously or in ascending order. Moreover, Sun provided the Hoffman-Wielandt-type theorem for generalized singular values of Grassman matrix pairs [13,14,15]. Xu et al. [16] also considered the constrained optimization problems of Grassman matrix pairs and they presented

$$\min_{U_kU_k^H=I_n,V_kV_k^H=I_m}|\det(cI_m\pm\prod_{k=1}^{s}\Gamma_kU_k\Delta_kV_k)|=\begin{cases}|c|^{m-r}\prod_{i=1}^{r}(|c|-\prod_{k=1}^{s}|\ddot{\gamma}_i^k||\ddot{\delta}_i^k|), & \prod_{k=1}^{s}|\ddot{\gamma}_r^k||\ddot{\delta}_r^k|\le|c|,\\ |c|^{m-r}\prod_{i=1}^{r}(\prod_{k=1}^{s}|\ddot{\gamma}_i^k||\ddot{\delta}_i^k|-|c|), & \prod_{k=1}^{s}|\ddot{\gamma}_1^k||\ddot{\delta}_1^k|\ge|c|,\\ 0, & otherwise,\end{cases}$$

$$\min_{U_kU_k^H=I_n,V_kV_k^H=I_m}|\mathrm{tr}(cI_m\pm\prod_{k=1}^{s}\Gamma_kU_k\Delta_kV_k)|=\begin{cases}m|c|-\sum_{i=1}^{r}\prod_{k=1}^{s}|\ddot{\gamma}_i^k||\ddot{\delta}_i^k|, & \frac{1}{m}\sum_{i=1}^{r}\prod_{k=1}^{s}|\ddot{\gamma}_i^k||\ddot{\delta}_i^k|\le|c|,\\ 0, & \frac{1}{m}\sum_{i=1}^{r}\prod_{k=1}^{s}|\ddot{\gamma}_i^k||\ddot{\delta}_i^k|\ge|c|,\end{cases}$$

where $r=\min\{m,n\}$, $\ddot{\gamma}_i^k$ and $\ddot{\delta}_i^k$ denote the i-th singular value of Γ_k and Δ_k, which are $m\times n$ and $n\times m$ complex matrices, respectively. Compared with the above special cases, (1.1) and (1.2) are more complicated because they involved more unitary constrained conditions. These motivated us to use new technique for giving the analytic solutions of (1.1) and (1.2).

1.2. Organization

The rest of this paper is organized as follows. In Section 2 we will give some notations and lemmas, which are useful to deduce the main results. In Section 3 we will provide the analytic solutions of (1.1) and (1.2). In Section 4 numerical examples are given to illustrate the theoretical results. Finally, concluding remarks are drawn in Section 5.

1.3. Notation

Throughout this paper we always use the following notations and definitions. Let $\mathbf{R}$, $\mathbf{C}$, $\mathbf{C}^{m\times n}$ and $\mathbf{U}_n$ be the sets of real numbers, complex numbers, $m\times n$ complex matrix set and $n\times n$ unitary matrices, respectively. $|\cdot|$ and $Re[\cdot]$ stand for absolute value and real part of a complex number, respectively. The symbols I_m and $O_{m\times n}$ stand for the identity matrix of order n and $m\times n$ zero matrix, respectively. For a matrix $\Gamma\in\mathbf{C}^{n\times n}$, $\det(\Gamma)$ and $\mathrm{tr}(\Gamma)$ denote the determinant and trace of the matrix Γ, respectively. We denote by $\sigma_i(\Gamma)$ the set of its singular values, and throughout the paper we assume that its singular values are arranged in decreasing order, i.e., $\sigma_1(\Gamma)\ge\sigma_2(\Gamma)\ge\cdots\ge\sigma_n(\Gamma)\ge 0$.

2. Preliminaries

Lemma 2.1 [1] Let $A_1,\cdots,A_m\in\mathbf{C}^{n\times n}$ with singular values $\sigma_1(A_j)\ge\sigma_2(A_j)\ge\cdots\ge\sigma_n(A_j),\ j=1,\cdots,m$