

New Perturbation Analysis of Generalized Singular Values of Grassmann Matrix Pairs with Arbitrary Permutation

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Abstract. In this paper we give new perturbation analysis of generalized singular values of Grassmann matrix pairs with arbitrary permutation from a new perspective. The new explicit expressions of the chordal metric between generalized singular values of Grassmann matrix pairs with arbitrary permutation are presented, which result in only two small-size singular value decompositions to evaluate. The proposed results are generalizations of several results on bounds on perturbation of generalized singular values.

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Key words: Perturbation analysis, Chordal metric, Generalized singular value, Grassman matrix pair, Arbitrary permutation.

1 Introduction

The generalized singular value decomposition (GSVD) for a matrix pair of two matrices with the same number of columns was proposed by Van Loan in 1976 [1]. Numerical methods and perturbation analysis of GSVD have been well developed, see [2–7]. Zha [8] proposed a generalized SVD for matrix triplets. Stewart [9] and Van Loan [10] proposed two algorithms for computing the GSVD. Sun [11] presented the Hoffman-Wielandt theorem for the generalized singular values (GSVs) of Grassman matrix pairs and gave bounds on perturbations of GSVs, which generalized several well-known results for the standard singular value problem. Li [12] presented several perturbation bounds of GSVs

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and its associated subspace. Xu et al. [13] provided the explicit expression and sharper bounds of the chordal metric between generalized singular values of Grassmann matrix pairs. We note that in these papers theoretical existence of chordal metric between generalized singular values of Grassmann matrix pairs with the permutation are discussed more.

In this paper, from a computable and practical perspective, we will analyze the calculation formula and model of the chordal metric between the generalized singular values of Grassmann matrix pairs with arbitrary permutations, and subsequently design an efficient algorithm. The proposed results can theoretically be considered generalizations of several findings regarding bounds on the perturbation of generalized singular values presented in [12, 14]. They can be further well applied to comparative analysis of gene mRNA expression data [15] and other aspects in applications.

1.1 Notation

Throughout this paper we always use the following notations and definitions. i denotes imaginary unit $\sqrt{-1}$. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, $\mathbb{C}^{m \times n}$ and $\mathbb{U}_n$ are the sets of real numbers, complex numbers, n -dimensional real vectors, $m \times n$ complex matrices and $n \times n$ unitary matrices, respectively. $|\cdot|$, $\Im(\cdot)$ and $\Re(\cdot)$ stand for absolute value, imaginary part and real part of a complex number, respectively. The symbols I_n and $O_{m \times n}$ stand for the identity matrix of order n and $m \times n$ zero matrix, respectively. For a square matrix $A \in \mathbb{C}^{n \times n}$, $\bar{A}$, A^T , A^H , A^{-1} , $\det(A)$, $\text{tr}(A)$ denote the conjugate, transpose, conjugate transpose, inverse, determinant and trace, respectively. By $\|\cdot\|_2$ we denote the spectral norm of a matrix, and by $\sin(\cdot)$ and $\cos(\cdot)$ we denote sin and cos functions, respectively. $e^{i\kappa}$ stands for $\cos\kappa + i\sin\kappa$ for angle κ . The singular value set of A is denoted by $\sigma(A)$. For given matrices $A, B \in \mathbb{C}^{n \times n}$, the notation $A < (\leq) B$ indicates that $B - A$ is a positive (semi-)definite matrix. The conjugate of a number $c \in \mathbb{C}$ is denoted by c^* . $\text{diag}(0, \dots, 0, 1_k, 0, \dots, 0) \in \mathbb{C}^{m \times m}$ denotes m -order diagonal matrices with the k th row and k th column element being 1 and the remaining main diagonal elements being zero. For a matrix $A \in \mathbb{C}^{n \times n}$, we denote the singular values by $\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_n(A) \geq 0$, arranged in decreasing order. We denote $a(b)$ by a or b . Let $X, Y \in \mathbb{C}^{m \times n}$ ($m > n$) both have full column rank n , and define the angle $\Theta(X, Y)$ between X and Y as [16]

$$\Theta(X, Y) \equiv \arccos((X^H X)^{-1/2} X^H Y (Y^H Y)^{-1} Y^H X (X^H X)^{-1/2})^{-1/2} \geq 0.$$

Definition 1.1. [17] Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{p \times n}$. A matrix pair $\{A, B\}$ is an (m, p, n) Grassman matrix pair (GMP) if $\text{rank}(A^T, B^T)^T = n$.

Definition 1.2. [17] Let $\{A, B\}$ be an (m, p, n) -GMP. A nonnegative number-pair (α, β) is a GSV of the GMP $\{A, B\}$ if

$$(\alpha, \beta) = (\sqrt{\lambda}, \sqrt{\mu}), \text{ where } (\lambda, \mu) \in \lambda(A^H A, B^H B) \text{ and } \lambda, \mu \geq 0.$$