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Heintze-Karcher Inequality for Anisotropic Free Boundary Hypersurfaces in Convex Domains

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Abstract. In this paper, we prove an optimal Heintze-Karcher-type inequality for anisotropic free boundary hypersurfaces in general convex domains. The equality is achieved for anisotropic free boundary Wulff shapes in a convex cone. As applications, we prove Alexandrov-type theorems in convex cones.

AMS subject classifications: 53C45, 53A10, 53C42

Key words: Heintze-Karcher's inequality, constant mean curvature, free boundary surface, capillary surface, convex cone.

1 Introduction

The Heintze-Karcher inequality states that for a bounded domain Ω in $\mathbb{R}^{n+1}$ with smooth and mean convex boundary $\Sigma = \partial\Omega$, it holds that

$$\int_\Sigma \frac{1}{H} \mathrm{d}A \geq \frac{n+1}{n} |\Omega|, \tag{1.1}$$

and equality in (1.1) holds if and only if Σ is a geodesic sphere. Here H is the mean curvature of Σ and Σ is said to be mean convex if $H > 0$. It was first proved by Heintze-Karcher [8] and in the present form (1.1) by Ros [17]. A combination of the Heintze-Karcher inequality (1.1) and the Minkowski-Hsiung formula yields Alexandrov's theorem for embedded closed constant mean curvature (CMC) hypersurfaces, namely any closed embedded CMC hypersurface is a sphere, first proved by Alexandrov [1] via moving plane method and by Reilly [16] and Ros [17] via integral method. For the history

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of the study of Heintze-Karcher's inequality and Alexandrov's theorem, we refer to [9] and the references therein. An anisotropic version of the Heintze-Karcher inequality has been proved by He-Li-Ma-Ge [7], which leads to an Alexandrov-type theorem for embedded closed constant anisotropic mean curvature hypersurfaces. Heintze-Karcher-type inequalities in space forms and more generally, in warped product manifolds, have been proved by Brendle [2] (see also [12]).

Recently, we have studied the Heintze-Karcher-type inequality for hypersurfaces with boundary in the half-space $\mathbb{R}^{n+1}_+=\{x\in\mathbb{R}^{n+1}|\langle x,E_{n+1}\rangle>0\}$. Precisely, we prove the following result.

Theorem 1.1 ([9, 10])**.** *Let $\Sigma\subset\overline{\mathbb{R}^{n+1}_+}$ be an embedded compact C^2-hypersurface with boundary $\partial\Sigma$ which intersects $\partial\mathbb{R}^{n+1}_+$ transversally such that $\langle\nu_F(x),-E_{n+1}\rangle\le 0$ for any $x\in\partial\Sigma$. Let Ω denote the domain enclosed by Σ and $\partial\mathbb{R}^{n+1}_+$. Assume the anisotropic mean curvature H^F of Σ is positive. Then we have*

$$\int_\Sigma \frac{F(\nu)}{H^F}dA\ge\frac{n+1}{n}|\Omega|. \tag{1.2}$$

Moreover, equality in (1.2) *holds if and only if Σ is an anisotropic free boundary Wulff cap.*

A hypersurface Σ, which intersects $\partial\mathbb{R}^{n+1}_+$ transversally, is called anisotropic free boundary if $\langle\nu_F(x),-E_{n+1}\rangle=0$ for any $x\in\partial\Sigma$, where ν_F is the anisotropic normal of Σ. A Wulff cap is part of a Wulff shape. Regarding the anisotropy F and anisotropic quantities with a sub- or superscription F, we use the notations in [10]. A brief overview is also provided in Section 2.

Using Theorem 1.1, we have proved an Alexandrov-type theorem for anisotropic free boundary or anisotropic capillary hypersurfaces in $\mathbb{R}^{n+1}_+$. This Alexandrov-type theorem for isotropic capillary hypersurfaces in $\mathbb{R}^{n+1}_+$ was first proved by Wente [21] via moving plane method.

In this paper, we continue the study of Heintze-Karcher-type inequality in a more general setting, in particular, for hypersurfaces with boundary in general convex domains.

Let $\mathcal{K}$ be the class of all convex domains in $\mathbb{R}^{n+1}$. We emphasize that in this paper for each $K\in\mathcal{K}$, ∂K is not assumed to be C^1, in other words, ∂K may have singularity. In order not to make situations too complicated, we restrict us to consider a subset of $\mathcal{K}$, denoted by $\mathcal{K}_P$, which contains convex domains with milder singularities, which we will describe.

We first define the class of convex polytopes with non-empty interior by $\mathcal{P}$. Each element $P\in\mathcal{P}$ can be expressed

$$P=\bigcap_{j\in J}\{u_j\le 0\},$$

where J is a finite index set and u_j $(j\in J)$ are affine linear functions on $\mathbb{R}^{n+1}$, i.e., P is determined by a family of inequalities $u_j\le 0$ $(j\in J)$. The family is called *irredundant* if $P\ne\bigcap_{j\in J,j\ne l}\{u_j\le 0\}$ for any $l\in J$. We can and will always assume that P is irredundant. A