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The Boundary Case of the J-Flow

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Abstract. In this paper, we shall study the boundary case for the J-flow under certain geometric assumptions.

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1 Introduction

Let M be a compact manifold without boundary of complex dimension $n \geq 2$, carrying two Kähler forms χ and ω. The J-flow is

$$\frac{\partial \varphi}{\partial t} = c - \frac{n(\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^{n-1} \wedge \omega}{(\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^n}, \qquad \varphi(z,0) = \varphi_0(z), \tag{1.1}$$

where $\chi + \sqrt{-1}\partial\bar{\partial}\varphi_0 > 0$ and

$$c := \frac{n\int_M \chi^{n-1} \wedge \omega}{\int_M \chi^n}.$$

The J-flow was constructed to study J-equation

$$c\left(\chi + \sqrt{-1}\partial\bar{\partial}\varphi\right)^n = n\left(\chi + \sqrt{-1}\partial\bar{\partial}\varphi\right)^{n-1} \wedge \omega, \qquad \chi + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \tag{1.2}$$

which is a stationary state of the J-flow (1.1). Eq. (1.2) was independently discovered by Donaldson [5] and Chen [2] under different geometric backgrounds.

For smooth admissible solutions, Donaldson [5] proposed that it is a sufficient condition that

$$[nc\chi - \omega] > 0, \tag{1.3}$$

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which was confirmed by Chen [2] for Kähler surfaces. Indeed, Eq. (1.2) can be rewritten as a complex Monge-Ampère equation on Kähler surfaces, which was solved by Yau [19]. However, Condition (1.3) does not work well in higher dimensions. Chen [3] proved the long time smoothness and existence of the J-flow. Later, a new condition was proposed by Song and Weinkove [12], that is, there exists a Kähler form $\chi' \in [\chi]$ such that

$$c\chi'^{n-1} - (n-1)\chi'^{n-2} \wedge \omega > 0. \tag{1.4}$$

Song and Weinkove [12] proved that Condition (1.4) is sufficient and necessary for the existence of smooth admissible solution to Eq. (1.2) by proving the convergence for the J-flow (1.1). Later, Fang, Lai and Ma [8] extended the results by parabolic flows to complex Monge-Ampère type equations

$$c(\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^n = n(\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^m \wedge \omega^{n-m}, \qquad \chi + \sqrt{-1}\partial\bar{\partial}\varphi > 0,$$

and defined the sufficient and necessary solvability condition as *cone condition*

$$c\chi'^{n-1} - m\chi'^{m-1} \wedge \omega^{n-m} > 0, \qquad \text{for some } \chi' \in [\chi]. \tag{1.5}$$

When Condition (1.4) degenerates to the boundary case

$$c\chi'^{n-1} - (n-1)\chi'^{n-2} \wedge \omega \geq 0, \tag{1.6}$$

it is interesting to study the behavior of the J-flow (1.1). On Kähler surfaces, Fang, Lai, Song and Weinkove [9] adapted the trick of Chen [2], and solved

$$(c\chi - \omega + \sqrt{-1}\partial\bar{\partial}u)^2 = \omega^2. \tag{1.7}$$

According to (1.6), $[c\chi - \omega]$ is semipositive and big. Eyssidieux, Guedj, and Zeriahi [7] showed that there is a bounded pluripotential solution to Eq. (1.7), which is smooth in $Amp(c\chi - \omega)$. Taking advantage of the L^∞ estimate for (1.7), Fang, Lai, Song and Weinkove [9] proved the t-independent C^0 estimate, and then studied the behavior of the solution. In this paper, we also derive the C^0 estimate via this way.

To study the boundary case, we need to impose more geometric conditions. Instead of Eq. (1.1), we study more general gradient flows,

$$\frac{\partial\varphi}{\partial t} = c - \frac{n(\chi + \tilde{\chi} + \sqrt{-1}\partial\bar{\partial}\varphi)^m \wedge \omega^{n-m}}{(\chi + \tilde{\chi} + \sqrt{-1}\partial\bar{\partial}\varphi)^n}, \qquad \varphi(z,0) = \varphi_0(z) \in \mathcal{H}, \tag{1.8}$$

where $1 \leq m < n$, $\mathcal{H} := \{u \in C^\infty(M) | \chi + \tilde{\chi} + \sqrt{-1}\partial\bar{\partial}u > 0\}$ and c in (1.8) is defined by

$$c := \frac{n\int_M (\chi + \tilde{\chi})^m \wedge \omega^{n-m}}{\int_M (\chi + \tilde{\chi})^n} > 0.$$

We impose the condition that $\tilde{\chi}$ is semipositve and big, and the boundary case of cone condition (1.5)

$$c\chi^{n-1} - m\chi^{m-1} \wedge \omega^{n-m} \geq 0. \tag{1.9}$$