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Global Existence of a Mean Curvature Flow in a Cone

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Abstract. We consider a mean curvature flow in a cone, that is, a hypersurface in a cone which propagates toward the opening of the cone with normal velocity depending on its mean curvature. In addition, the contact angle between the hypersurface and the cone boundary depending on its position. First, we construct a family of radially symmetric self-similar solutions. Then we use these solutions to give a priori estimates for the solutions of the initial boundary value problems, and show their global existence.

AMS subject classifications: 35K93, 53E10, 53A05, 35R35

Key words: Mean curvature flow, quasilinear parabolic equation, free boundary problem, self-similar solution.

1 Introduction

We consider the propagation of a hypersurface in a cone. The law of the motion of the hypersurface is the so-called mean curvature flow

$$V = H \quad \text{on } \Gamma_t \subset \Omega, \tag{1.1}$$

where, Γ_t denotes a time-dependent hypersurface in a cone Ω, Γ_t contacts the boundary of Ω with prescribed angles, V and H denote the normal velocity and the mean curvature of Γ_t, respectively, and the cone Ω is defined as

$$\Omega := \left\{ (x,y) \in \mathbb{R}^{N+1} \mid y \in \mathbb{R} \text{ satisfies } y > |x|,\ x = (x_1,\cdots,x_N) \in \mathbb{R}^N \right\}.$$

Mean curvature flow (1.1) as well as its generalized versions have been extensively studied in the last decades. To name only a few, Gage and Hamilton [9], Grayson [10,11],

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Angenent [1,2], Chou and Zhu [7] and references therein considered shrinking closed plane curves driven by (1.1). Huisken [14,15] etc. considered closed surfaces in higher dimension spaces. On the other hand, the mean curvature flows (with or without driving forces) in domains with boundaries were considered by some authors. For example, in the case where Ω is a cylinder, it was studied by Altschuler and Wu [3,4], Matano, Lou, et al. [5,19–21] etc. Under certain conditions, it was shown that the flow will converge to a traveling wave (or, translating solution); In the case where Ω is the half space or a sector on the plane, the existence and asymptotic behavior of the flow were studied in [6,8,12,13,16,17,22] etc. under the boundary condition: Γ_t contacts the boundary of Ω with constant angles.

In this paper we consider graphic surfaces in the cone Ω, that is, for some function $y=u(x,t)$,

$$\Gamma_t=\{(x,u(x,t))\,|\,x\in\omega(t)\subset\mathbb{R}^N\}\subset\Omega, \tag{1.2}$$

where $\omega(t)$ is the definition domain of u containing the origin. To avoid sign confusion, the unit normal vector $\mathbf{n}$ to Γ_t will always be chosen upward, and so

$$\mathbf{n}=\frac{(-Du,1)}{\sqrt{1+|Du|^2}} \qquad \text{and} \qquad V=\frac{\partial}{\partial t}\Gamma_t\cdot\mathbf{n}=\frac{u_t}{\sqrt{1+|Du|^2}}.$$

The sign of H will be understood in accordance with this choice of the direction of the normal, which means that H is positive at those points where the hypersurface is convex. So,

$$H=-\mathrm{div}_{(x,y)}\mathbf{n}=\mathrm{div}_x\left[\frac{Du}{\sqrt{1+|Du|^2}}\right]=\Big(\delta_{ij}-\frac{D_iuD_ju}{1+|Du|^2}\Big)\frac{D_{ij}u}{\sqrt{1+|Du|^2}}.$$

Thus, the mean curvature flow (1.1) is expressed as

$$u_t=\Big(\delta_{ij}-\frac{D_iuD_ju}{1+|Du|^2}\Big)D_{ij}u, \qquad x\in\omega(t)\subset\mathbb{R}^N,\ t>0. \tag{1.3}$$

In addition, we require that Γ_t contacts $\partial\Omega$, the boundary of Ω, on the closed curve $\{(x,|x|)\,|\,x\in\partial\omega(t)\}$ with prescribed angle $\phi\in(0,\frac{\pi}{4})$. Denoting by $\nu:=\frac{1}{\sqrt{2}|x|}(-x,|x|)$ the inner unit normal to $\partial\Omega$ at point $(x,|x|)$, we obtain the following boundary condition to our problem:

$$\mathbf{n}\cdot\nu=\frac{x\cdot Du+|x|}{|x|\sqrt{2(1+|Du|^2)}}=\cos\phi. \tag{1.4}$$

Therefore, our problem can be expressed by the quasilinear parabolic equation (1.3) with oblique boundary condition (1.4).

In the special case where Γ_t is a radially symmetric (with respect to y-axis) surface, we have $u(x,t)=u(r,t)$ with $r=|x|$, and so $\omega(t)=B_{\xi(t)}(0):=\{x\in\mathbb{R}^N\,|\,|x|<\xi(t)\}$ for some $\xi(t)$ satisfying $\xi(t)=u(\xi(t),t)$. In this case, Eq. (1.3) is reduced to

$$u_t=\frac{u_{rr}}{1+u_r^2}+\frac{(N-1)u_r}{r}, \qquad 0<r<\xi(t),\ t>0, \tag{1.5}$$