J. Math. Study
doi: 10.4208/jms.v57n3.24.05

Vol. **57**, No. 3, pp. 309-330
September 2024

Well-Posedness of a Nonlocal Stage-Structured Population Model with Free Boundary§

Jian Fang[1,*] and Kaiyuan Tan[2]

[1] *Institute for Advanced Studies in Mathematics and School of Mathematics, Harbin Institute of Technology, Harbin 150001, China;*
[2] *School of Mathematics, Harbin Institute of Technology, Harbin 150001, China.*

Received April 4, 2024; Accepted July 11, 2024;
Published online September 26, 2024.

Abstract. We propose a delay-induced nonlocal free boundary problem by modeling the invasion of a two-stage structured species, where the nonlocal interaction is caused jointly by time delay, free boundary and the diffusion. After establishing a comparison principle and a priori boundedness estimates, we prove the local and global existence of classical solutions for the model.

AMS subject classifications: 35K57, 35R35, 34K05, 92D25

Key words: Reaction-diffusion equation, free boundary, time delay.

---

1 Introduction

This paper is concerned with the following nonlocal problem with Stefan type free boundary conditions:

$$\begin{cases} u_t = A(t,x)u_{xx} + B(t,x)u_x + C(t,x,u,v(\tau(t),x;t)), & t>0,\ x\in(g(t),h(t)),\\ u(t,g(t))=0,\ g'(t)=-\mu u_x(t,g(t)), & t>0,\\ u(t,h(t))=0,\ h'(t)=-\mu u_x(t,h(t)), & t>0,\\ u(\theta,x)=\phi(\theta,x), & \theta\in[-\tau(0),0],\ x\in[g(\theta),h(\theta)], \end{cases} \tag{P}$$

where $v(\tau(t),x;t)$ is the solution $v(s,x)$, evaluated at $s=\tau(t)$, of following problem

$$\begin{cases} v_s = E(s,x)v_{xx} + F(s,x)v_x + G(s,x)v, & s\in(0,\tau(t)],\ x\in(g(s+t-\tau(t)),h(s+t-\tau(t))),\\ v(s,x)=0, & s\in(0,\tau(t)],\ x=g(s+t-\tau(t)) \text{ or } h(s+t-\tau(t)),\\ v(0,x)=f(t-\tau(t),u(t-\tau(t),x)), & x\in[g(t-\tau(t)),h(t-\tau(t))]. \end{cases} \tag{Q}$$

*Corresponding author. *Email addresses:* `jfang@hit.edu.cn` (Fang J), `21B912022@stu.hit.edu.cn` (Tan K)

Here A,B,C,E,F,G,f,τ are given functions that will be specified later.

This model is used to describe the population invasion of an alien species, which is classified by age into mature stage u and immature stage v; an individual at time t belongs to immature stage if and only if its age does not exceed a time-dependent number $\tau(t)$, where $\tau(t)$ is often called the maturation period. As such, $v(\tau(t),x;t)$ is the newly added mature population at time t, which is an evolution result of newborns $f(t-\tau(t),u(t-\tau(t),x))$ at time $t-\tau(t)$. Here f is the birth function of mature population. In the model, we also assume that the mature population is responsible for the expansion of the habitat $(g(t),h(t))$ by obeying the Stefan type free boundary conditions, while the immature population v moves inside this habitat but does not contribute to the habitat expansion. Thus, the model is highly nonlocal in space and time, involving also the to-be-determined habitat over the time interval $(t-\tau(t),t)$. The derivation details of this model will be presented in the next section.

The model (P)-(Q) is closely related to some classical models in literature. If we choose $\tau(t)\equiv 0$ and

$$A\equiv 1,\quad B\equiv 0,\quad C=-u+v$$

in (P)-(Q), then it becomes the following problem:

$$\begin{cases} u_t=u_{xx}-u+f(u), & t>0,\ x\in(g(t),h(t)),\\ u(t,g(t))=0,\ g'(t)=-\mu u_x(t,g(t)), & t>0,\\ u(t,h(t))=0,\ h'(t)=-\mu u_x(t,h(t)), & t>0,\\ u(0,x)=\phi(0,x), & x\in[g(0),h(0)], \end{cases} \tag{1.1}$$

which was proposed by Du and Lin [7] in 2010. With a KPP type setting, they revealed a spreading-vanishing dichotomy for the long-term behavior of solutions, which is different from the hair-trigger effect established in the earlier celebrated work [1] by Aronson and Weinberger in 1970's for the Cauchy problem of the KPP equation. We refer to [2] for an interpretation of the Stefan type free boundary conditions in population ecology.

In the past decade, there have been various extensions of the work [7] in several directions. For instance, Du and Lou [9] gave a rather complete description of the long-term behavior of solutions for (1.1) with general nonlinearities. Du, Guo and Peng [6] considered a time periodic case in radially symmetric higher dimension space. Wang [24] extended it to the case where the intrinsic growth rate is sign-changing in time, by a delicate application of principle eigenvalues. Gu, Lin and Lou [12] incorporated advection into the model and found different effects of large and small advection, see also [13, 22]. There also have been an increasing interest in extensions to systems, including competition models and predator-prey models, see for example [3, 8, 10, 14, 17, 26, 27, 29] and references therein.

Based on the framework of [7], Du, Fang and Sun [5] incorporated time delay into the