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Transition Layers to Chemotaxis-Consumption Models with Volume-Filling Effect

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Abstract. We are interested in the dynamical behaviors of solutions to a parabolic-parabolic chemotaxis-consumption model with a volume-filling effect on a bounded interval, where the physical no-flux boundary condition for the bacteria and mixed Dirichlet-Neumann boundary condition for the oxygen are prescribed. By taking a continuity argument, we first show that the model admits a unique nonconstant steady state. Then we use Helly's compactness theorem to show that the asymptotic profile of steady state is a transition layer as the chemotactic coefficient goes to infinity. Finally, based on the energy method along with a cancellation structure of the model, we show that the steady state is nonlinearly stable under appropriate perturbations. Moreover, we do not need any assumption on the parameters in showing the stability of steady state.

AMS subject classifications: 35Q92, 35B35, 92C17

Key words: Chemotaxis, volume-filling, transition layer, stability, physical boundary conditions.

1 Introduction

In order to prevent the cell overcrowding that is undesirable from a biological standpoint, Hillen and Painter [4,19] proposed the following chemotaxis model with a volume-filling effect:

$$\begin{cases} u_t = \nabla\cdot[(q(u)-q'(u)u)\nabla u - \chi q(u)u\nabla v] + f(u,v), \\ v_t = \Delta v + g(u,v), \end{cases} \tag{1.1}$$

where u is the density of cell population, and v is the concentration of chemical signal. The positive parameter χ represents the chemotactic coefficient that measures the strength of chemoattractants. $q(u)$ denotes the probability of cells finding space at their

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neighboring locations, which generally satisfies the following condition: there is a maximal cell number K, called crowing capacity, such that

$$q(u)\equiv 0, \qquad \forall u\geq K \qquad \text{and} \qquad 0<q(u)\leq 1, \quad q'(u)\leq 0, \qquad \forall\, 0\leq u<K.$$

$f(u,v)$ and $g(u,v)$ represent the cell and chemical kinetics, respectively. Typical candidates of $g(u,v)$ include $g(u,v)=u-v$ and $g(u,v)=-uv$. In the former case, the signal is produced by the cells, and the system is a chemotaxis-production model. In the latter case the signal is consumed by the cells, and the system is a chemotaxis-consumption model. In contrast to the classical chemotaxis model introduced by Keller and Segel [8], the system (1.1) does not treat cells as point masses but takes into account the finite size of cells.

There are fruitful analytical works for the system (1.1) when the chemical signal is produced by the cells, i.e., $g(u,v)=u-v$. If there is no growth term, i.e., $f(u,v)=0$, the system is relatively easy to handle and there are many profound results. In the case $q(u)=1-u$, Hillen and Painter [4] first proved the global existence of classical solutions on a compact Riemannian manifold without boundary. Wrzosek [29] generalized the works of [4] to weak solutions on a bounded domain with general boundary conditions, and showed that, there is a global attractor by using the dynamical system approach. Wrzosek [30] further studied the large time behaviors of solutions by constructing Lyapunov functional. Subsequently, Jiang and Zhang [7] showed that every solution of the system converges to an equilibrium as the time goes to infinity by using a non-smooth version of the Łojasiewicz-Simon inequality. Wang *et al.* [27,28] further considered the general volume-filling effect, and discovered a critical parameter condition to separate the global existence and formation of singularity, where the singularity means the solution attains the value 1 in either finite or infinite time. The volume-filling mechanism excludes the blow-up behaviors of cell density but usually generates complex biological patterns. Potapov and Hillen [20] performed the local bifurcation analysis to investigate the pattern formation in 1D. Wang and Xu [25] generalized the work of [20] through a global bifurcation analysis. In fact, the global bifurcation theory was initially used in [24] to prove the existence of spiky solutions of the chemotaxis models. Moreover, [25] also showed that, in contrast to the classical Keller-Segel model where the one is a spike, the asymptotic profile of steady state of the system (1.1) with $g(u,v)=u-v$, $q(u)=1-u$ and $f(u,v)=0$ is indeed a transition layer in the sense that the limit of cell density is a step function as $\chi\to\infty$. If the logistic growth of cells is taken into account, i.e. $f(u,v)=u(1-u)$, the system is very difficult to analyze and there are only a few results achieved. Ma *et al.* [15] employed the index theory together with the maximum principle to show that under the effect of growth, the system has nonconstant steady states as χ grows. Ma *et al.* [14,16,17] further generalized the works of [15] to models with general volume-filling effect. Furthermore, [17,19,26] numerically predicted that under the effect of growth, the system generates spatio-temporal patterns and even chaotic patterns. Besides the spatial patterns, Ou and Yuan [18] proved the existence of traveling front if the chemotactic coefficient is small. Although the system admits various patterns, it is usually challenging