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Fundamental Groups of Manifolds of Positive Sectional Curvature and Bounded Covering Geometry

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Abstract. Let M be an n-manifold of positive sectional curvature ≥ 1. In this paper, we show that if the Riemannian universal covering has volume at least $v>0$, then the fundamental group $\pi_1(M)$ has a cyclic subgroup of index bounded above by a constant depending only on n and v.

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1 Introduction

In this paper, we study the fundamental group of a complete (odd-dimensional) manifold M of positive sectional curvature.

If M is not compact, then M is diffeomorphic to an Euclidean space ([16]), and if M is compact (closed without boundary), then the fundamental group of M is finite (Bonnet theorem), and the Synge theorem asserts that for $n=\dim(M)$ even, M is simply connected or M is not orientable, and for n odd, M is orientable. Indeed, in each odd-dimension, the spherical space forms contain infinitely many fundamental groups ([40]).

A core problem in Riemannian geometry has been finding a topological obstruction on a compact (simply connected) manifold of non-negative sectional curvature which does not support a metric of positive sectional curvature. By Sygne theorem, given each $2n\geq 4$, the $2n$-manifold which is the metric product of two real projective spaces supports no metric of positive sectional curvature; while the Hopf conjecture has been open, which asserts that the product of two unit spheres admits no metric of positive sectional curvature.

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In view of the above, a natural problem is to find, in odd dimensions, a constraint on the structure of fundamental groups.

Conjecture 1.1 ($c(n)$-cyclicity, [32]). *Let M be a compact n-manifold of positive sectional curvature. Then the fundamental group of M is $c(n)$-cyclic i.e., $\pi_1(M)$ contains a normal cyclic subgroup of index $\le c(n)$, a constant depending only on n.*

In Conjecture 1.1, a bound on index depending on n is of necessary (e.g., fundamental groups of spherical n-forms, [40]). Conjecture 1.1 proposes a dimension-related structural constraint on a fundamental group; e.g., Conjecture 1.1 implies that for any integer $h > c(9)$, the metric product of spherical space forms, $M = S^3_1 \times (S^3_1/\mathbb{Z}_h) \times (S^3_1/\mathbb{Z}_h)$ (orientable), admits no metric of positive sectional curvature.

Conjecture 1.1 was partially motivated by a result in [32] (see Theorem 1.1 below), which was from an early attempt to the problem of Chern [10]: if any abelian subgroup of the fundamental group is cyclic (an analogue of Preissmann theorem on a compact manifold of negative sectional curvature). Negative answers have been found in dimensions 7 [18, 35] and 13 [1]; the only odd dimensions in which (infinitely many) simply connected compact manifolds of positive sectional curvature are constructed.

Theorem 1.1. *Let M be a compact n-manifold of positive sectional curvature, $sec_M \ge 1$. Conjecture* 1.1 *holds for either of the following classes of manifolds:*

($S1.1$) *[33, 38] The isometry group of the Riemannian universal covering space, $\pi : \tilde{M} \to M$, has a positive dimension.*

($S1.2$) *[32, 33] M satisfies that $sec_M \le K$ and its volume $vol(M) < \epsilon(n)$, a small constant depending only on n.*

Note that (S1.1) was obtained in [33] by assuming that $\tilde{M}$ admits an isometric torus T^k-action ($k \ge 1$) whose orbits are preserved by the $\pi_1(M)$-action, and the $\pi_1(M)$-invariance is always satisfied for a subgroup of $\pi_1(M)$ of index bounded above by a constant depending only on n ([38]).

A proof of (S1.1) relies on the Synge theorem and the total Betti number estimate in [17]: without loss of generality, we may assume that M admits an isometric S^1-action, and $\pi_1(M)$ is not cyclic. By a Synge type argument, there is a subgroup $\mathbb{Z}_p < S^1$ (p is a prime) with fixed point set, $F(M,\mathbb{Z}_p) = \bigcup_{j=0}^{k-1} F_j \ne \emptyset$ (F_j denotes a component), such that $S^1|_{F_0}$ is not trivial. This allows one to apply induction on dimension to (F_0, S^1), and conclude that the subgroup, $\Lambda = \mathrm{Im}[\pi_1(F_0) \to \pi_1(M)]$, has a cyclic subgroup of index $c_1(n)$. It is not hard to show that the index, $[\pi_1(M):\Lambda] \le kl$, where l denotes the maximal number of components of $\pi^{-1}(F_j)$ ($0 \le j \le k-1$), and $k \le \mathrm{rank}(H_*(M,\mathbb{Z}_p))$ and $l \le \mathrm{rank}(H_*(M;\mathbb{Z}_p))$ ([2]), which is bounded above by a constant $c_2(n)$ ([17]).

A proof of (S1.2) is to show that M admits a nearby metric satisfying (S1.1), based on the singular nilpotent fibration theorem on a collapsed manifold with bounded sectional