

Notes on Conformal Metrics of Negative Curvature on Manifolds with Boundary

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Abstract. We use certain Morse functions to construct conformal metrics such that the eigenvalue vector of modified Schouten tensor belongs to a given cone. As a result, we prove that any Riemannian metric on compact 3-manifolds with boundary is conformal to a compact metric of negative sectional curvature.

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1 Introduction

In Riemannian geometry, a basic problem is to find a metric so that the various curvatures satisfy prescribed properties. A well-known result on this direction is the existence of metrics with negative Ricci curvature. In [3] Gao-Yau proved that any closed 3-manifold admits a Riemannian metric with negative Ricci curvature. Subsequently, Gao-Yau's theorem was generalized by Lohkamp [9, 10] to higher dimensions. In addition, Lohkamp's result allows manifolds with boundary.

This note is devoted to constructing conformal metrics with certain restrictions to curvatures. Let K_g , Ric_g and R_g denote sectional, Ricci and scalar curvature of the Riemannian metric g , respectively, with respect to the Levi-Civita connection.

Our first result is concerned with sectional curvature of 3-manifolds.

Theorem 1.1. *Let $(\bar{M}, g)$ be a three dimensional compact connected Riemannian manifold with smooth boundary ∂M , $\bar{M} = M \cup \partial M$. Henceforth, M denotes the interior of $\bar{M}$. Then there is a smooth compact conformal metric $g_u = e^{2u}g$ of negative sectional curvature.*

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We shall mention that this result is related to main result of Hass [7][†], where he constructed on every 3-manifold with boundary a metric such that with respect to the metric the manifold has negative sectional curvature and the boundary is concave outwards. Since our construction operates effectively within every conformal class, Theorem 1.1 is notably distinct and different from others.

Our proof is also fairly different. It is based on Morse theory, as well as the relation between sectional curvature and the Einstein tensor in dimension three.

Lemma 1.1. Fix $x \in M^3$, let $\Sigma \subset T_x M$ be a tangent 2-plane, $\vec{n} \in T_x M$ the unit normal vector to Σ , then

$$G_g(\vec{n}, \vec{n}) = -K_g(\Sigma),$$

here

$$G_g := Ric_g - \frac{R_g}{2}g$$

stands for the Einstein tensor of g .

Despite Cartan-Hadamard theorem and the complexity of topology of underlining manifolds, in general, one could not expect that the resulting metric is complete. A nice complement to this topological obstruction is due to Gursky-Streets-Warren [5]. They proved that any compact Riemannian 3-manifold with smooth boundary admits a complete conformal metric of "almost negative" sectional curvature.

In fact we prove more general results than Theorem 1.1. Let Γ be an open, symmetric, convex cone in $\mathbb{R}^n$ with vertex at the origin, $\partial\Gamma \neq \emptyset$, and

$$\Gamma_n := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \text{each } \lambda_i > 0\} \subseteq \Gamma.$$

Following [2], Γ is a type 1 cone if $(0, \dots, 0, 1) \in \partial\Gamma$; otherwise, Γ is a type 2 cone. As in [13], for Γ we denote q_Γ the constant with

$$(1, \dots, 1, 1 - q_\Gamma) \in \partial\Gamma. \quad (1.1)$$

It is easy to see $1 \leq q_\Gamma \leq n$. In particular, for the k -th Gårding cone Γ_k , $q_{\Gamma_k} = \frac{n}{k}$.

For $n \geq 3$ we denote the modified Schouten tensor by (see [6])

$$A_g^\tau = \frac{1}{n-2} \left(Ric_g - \frac{\tau}{2(n-1)} R_g \cdot g \right), \quad \tau \in \mathbb{R}.$$

For $\tau = n-1$, it corresponds to the Einstein tensor. When $\tau = 1$, it is the Schouten tensor

$$A_g = \frac{1}{n-2} \left(Ric_g - \frac{1}{2(n-1)} R_g \cdot g \right).$$

For simplicity, we denote $A_g^{\tau, \alpha} = \alpha A_g^\tau$ with $\alpha = \pm 1$.

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