

Some Topics in the Ricci Flow

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Abstract. The Ricci flow plays an essential role in modern geometric analysis. In this short note, we only survey some special topics of this broad and deep field. We first survey some convergence results of the Ricci flow and the Kähler Ricci flow. In particular, we explain the basic idea in the proof of the Hamilton-Tian conjecture. Then we survey the recent progresses on the extension conjecture, which predicts that the Ricci flow can be extended when scalar curvature is bounded.

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1 Introduction to the Ricci flow

In 1982, R. Hamilton [31] introduced the Ricci flow equation

$$\frac{\partial}{\partial t}g = -2Ric, \quad (1.1)$$

which deforms the metric in the direction of negative Ricci curvature, aiming to find Einstein metric from any Riemannian metric via a canonical path. On a closed 3-manifold, he proved that any metric with positive Ricci curvature can be deformed into a space form metric via the normalized Ricci flow:

$$\frac{\partial}{\partial t}g = -2Ric + \frac{2}{n}rg, \quad r = \frac{\int_M R dv}{Vol(M, g)}. \quad (1.2)$$

This certainly raises the hope that one can use the Ricci flow to attack Poincaré conjecture. This hope was realized by G. Perelman in 2002, through his breakthrough work [27–29]. Since then, the Ricci flow has attracted intensive attentions. The last two decades

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witnessed huge advance in this field. For instance, Brendle-Schoen used the Ricci flow to prove the sphere theorem [6], Chen-Wang proved the Hamilton-Tian conjecture [16, 17] and Chen-Sun-Wang obtained a Ricci flow proof of Yau's stability conjecture [12], Bamler-Kleiner provided a Ricci flow proof of the generalized Smale conjecture [3].

Along the unnormalized Ricci flow equation, direct calculation shows the evolution equations of different quantities:

$$\frac{\partial R}{\partial t} = \Delta R + 2|Ric|^2, \quad (1.3)$$

$$\frac{\partial R_{ij}}{\partial t} = \Delta R_{ij} + 2R_{ikjl}R_{kl} - 2R_{ik}R_{kj}, \quad (1.4)$$

$$\begin{aligned} \frac{\partial R_{ijkl}}{\partial t} = & \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ & - (R_{ip}R_{pjkl} + R_{jp}R_{ipkl} + R_{kp}R_{ijpl} + R_{lp}R_{ijkp}), \end{aligned} \quad (1.5)$$

where $B_{ijkl} \triangleq -R_{ipqj}R_{kpql}$. From the above evolution equations, we know that the positivity of the scalar curvature and curvature operator are preserved along the flow. However, the positivity of the Ricci curvature is not preserved except in dimension 2 and 3.

Traditionally, the study of Ricci flow focuses on the application of the maximum principle. After the landmark work of Perelman [29], the entropy method starts to play the key role. In this celebrated paper, Perelman introduced his famous entropy $\mathcal{F}$ and $\mathcal{W}$. For a given closed Riemannian manifold (M, g) and a smooth function f satisfying $\int_M e^{-f} dv = 1$, the entropy $\mathcal{F}(g, f)$ is defined as

$$\int_M \{R + |\nabla f|^2\} e^{-f} dv.$$

The functional $\lambda(M, g)$ is defined as the infimum of $\mathcal{F}(g, f)$ among all smooth function f satisfying $\int_M e^{-f} dv = 1$. Rewrite $\varphi = e^{-\frac{f}{2}}$. Then it is clear that

$$\lambda(g) = \inf_{\int_M \varphi^2 dv = 1} \int_M \varphi(-4\Delta + R)\varphi dv.$$

Namely, $\mathcal{F}$ is the first eigenvalue of the operator $-4\Delta + R$. If $g(t)$ satisfies the Ricci flow equation, then

$$\frac{d}{dt} \lambda(g(t)) = 2 \int_M |R_{ij} + f_{ij}|^2 e^{-f} dv \geq 0.$$

Perelman shows that the Ricci flow is the gradient flow of the $\mathcal{F}$ -functional in the moduli space of Riemannian metrics. This property rules out circling flow lines and makes the formal dynamical picture of the Ricci flow relatively easy. If the Ricci flow has global existence, then it must converge to a critical point of the $\mathcal{F}$ -functional, which is a gradient steady soliton.