

A Survey of Classification Results for Solutions to the Liouville Equation

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Abstract. In this expository paper, we survey some results concerning the classification of solutions to the Liouville equation $\Delta u + e^{2u} = 0$ in $\mathbb{R}^2$.

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1 Introduction

The Liouville equation in dimension two

$$\Delta u + e^{2u} = 0 \tag{1.1}$$

is probably one of the oldest partial differential equations which continuously stimulates a great amount of researches till nowadays. Historically, it was Monge who first derived this equation in his study of prescribing constant Gauss curvature problem. Liouville obtained his representation formula (2.1) in 1853 [73], and Eq. (1.1) beared the name of Liouville shortly afterwards. We recommend [7, Section I.A] and [11] for a more detailed historical account of Monge's problem and Liouville's formula (see Theorem 2.1).

The Liouville equation is a semilinear elliptic equation with exponential nonlinearity. Equations of this type have been studied extensively both from mathematics and physics. Aside from its intimate relation with prescribing Gauss curvature problem under conformal change of metrics [21, 29, 32, 34, 38, 47], the Liouville type equation appeared naturally in the context of Mose-Trudinger inequalities [6, 40, 82, 87, 95], the mean field equation [12, 13, 15, 16, 18, 19, 22, 25, 31, 63, 71, 76], Toda systems [61, 77, 94, 98]. It also arises in physical processes such as statistical mechanics of two-dimensional turbulence [28, 63], gravitational equilibrium of polytropic stars [20], membrane buckling [37], combustion

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of a mixture of gases [8], thermo-ionic emission [90], string theory [89], Chern-Simons theory and mean field theory [39, 42, 86, 94].

Clearly, a comprehensive survey on the Liouville equation would be a very sophisticated mission which we are not aiming for. Instead, we mainly focus on the classification results of the Liouville equation, and present various proofs of these classification results to the best of the authors' knowledge. We recommend survey articles [73, 80, 81] from different viewpoints regarding the Liouville equation.

The geometric meaning of (1.1) is as follows. Let (M, g_0) be a Riemann surface, suppose $g = e^{2u}g_0$ is a Riemannian metric conformal to g_0 , the transformation law for the Gauss curvature under the conformal transformation is

$$\Delta u + K(x)e^{2u} = K_0(x), \quad (1.2)$$

where K, K_0 stand for the Gauss curvature of g and g_0 respectively. Therefore, geometrically, a solution of (1.1) gives rise to a conformal metric $g = e^{2u}g_0$ with constant Gaussian curvature $K(x) \equiv 1$.

In higher dimensions ($n \geq 3$), let $g = u^{\frac{4}{n-2}}g_0$ be a metric conformal to g_0 , then the scalar curvature R_g and R_0 of g and g_0 respectively are related by

$$\frac{4(n-1)}{n-2} \Delta u + R_g u^{\frac{n+2}{n-2}} = R_0 u.$$

We refer it as the scalar curvature equation. It is a semi-linear elliptic equation with critical Sobolev exponent. In the standard Euclidean space $(\mathbb{R}^n, g)$ with R_g being a positive constant, the above equation becomes

$$\Delta u + u^{\frac{n+2}{n-2}} = 0. \quad (1.3)$$

This equation can be viewed as the higher dimensional analogue of the Liouville equation. Gidas-Nirenberg [55] first established the radial symmetry of the entire solution of (1.3) under the assumption $u(x) \sim O(|x|^{2-n})$ as $|x| \rightarrow \infty$. In a remarkable paper [24], Caffarelli-Gidas-Spruck proved the radial symmetry of the solution without any assumption on the asymptotic behavior of u . Therefore as we shall see that the classification of entire solutions of (1.1) is more subtle than its higher dimensional analogue.

The paper is organized as follows. In Section 2, we present the Liouville's formula, which yields an explicit expression of the solution u in terms of its developing function f . In Section 3, we focus on the classification of finite total curvature solutions of (1.1). In Section 4, we discuss some other classification results to (1.1) under some other additional hypotheses. In Section 5, some geometric aspects of the Liouville equation are discussed.

2 Liouville's formula

In this section, we review the Liouville's formula and its generalizations.