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Two-Parameter Quantum Group Coming from Two-Parameter Deformed Virasoro Algebra of Hom-Type

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Abstract. In this paper, firstly, we use the bosonic oscillators to construct a two-parameter deformed Virasoro algebra, which is a non-multiplicative Hom-Lie algebra. Secondly, a non-trivial Hopf structure related to the two-parameter deformed Virasoro algebra is presented, that is, we construct a new two-parameter quantum group.

AMS subject classifications: 17A30, 17B61, 17B68

Key words: Hom-Lie algebra, bosonic oscillator, the two-parameter deformed Virasoro algebra, Hopf algebra.

1 Introduction

The q-deformed Virasoro algebras are given by many authors ([1,6,8,11,13,16,17]) respectively, which can be viewed as a typical examples of the physical applications of quantum group. Quantum groups are a kind of non-commutative and cocommutative Hopf algebras, which were introduced by Drinfeld and Jimbo as a q-deformation of the universal enveloping algebra of a Lie algebra [4,5,7,15]. Two-parameter quantum deformation is a generalization of the one-parameter quantum deformation. Two-parameter quantum enveloping algebras are known to have a generalized root space structure and the Drinfeld realizations of the two-parameter quantum enveloping algebras were studied in [12].

As a generalization of Lie algebras, Hom-Lie algebras were introduced by Hartwig, Larsson and Silvestrov in [8] as part of a study of deformations of the Witt and the Virasoro algebras. The motivations to study Hom-Lie structures are related to physics and

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to deformations of Lie algebras, in particular Lie algebras of vector field [2, 3, 6, 17]. A *Hom-Lie algebra* is a triple $(L,[\cdot,\cdot],\alpha)$. Here, L is a vector space, α is an endomorphism of L, and the skew-symmetric bracket satisfies the following conditions

$$[x,y]=-[y,x] \quad \text{(skew symmetry)}, \tag{1.1}$$
$$[\alpha(x),[y,z]]+[\alpha(y),[z,x]]+[\alpha(z),[x,y]]=0, \qquad \forall x,y,z\in L, \tag{1.2}$$

where (1.2) is generalized Jacobi identity.

Obviously, Lie algebras are special cases of Hom-Lie algebras in which α is the identity map.

In [6], Elchinger et. al introduced the two parameters deformed Virasoro algebra $V_{p,q}$, which is a Hom-Lie algebra. Concretely, $V_{p,q}=(\hat{L},\hat{\alpha})$ has basis $\{L_n,C|n\in\mathbb{Z}\}$ and bracket relations:

$$[L_n,L_m]:=(\frac{[n]}{p^n}-\frac{[m]}{p^m})L_{n+m}+\delta_{m+n,0}\frac{(q/p)^{-n}}{6(1+(q/p)^n)}\frac{[n-1]}{p^{n-1}}\frac{[n]}{p^n}\frac{[n+1]}{p^{n+1}}C,$$
$$[\hat{L},C]:=0,$$

and $\hat{\alpha}:\hat{L}\longrightarrow\hat{L}$ is the endomorphism of $\hat{L}$ defined by $\hat{\alpha}(L_n)=((1+(q/p)^n))L_n$, $\hat{\alpha}(C)=C$. The main tools of realizing the Hom-Lie algebra $V_{p,q}$ are (σ,τ)-derivations which are generalized derivations twisting the Leibniz rule by means of a linear map.

In 1998, Hu gave the quantum group structure of the q-deformed Virasoro algebra in [10]. In [2], Cheng and Su developed an approach to construct a q-deformed Heisenberg-Virasoro algebra, which is a Hom-Lie algebra, and the quantum deformations of Heisenberg-Virasoro algebra which provided a nontrivial Hopf structure were presented. In [18], Yuan realized the q-deformation $W(2,2)$ by using the bosonic and fermionic oscillators in physics, the quantum group structure of q-deformation on Lie algebra $W(2,2)$ is further determined. For the supervision, a two-parameter quantum deformation of Lie superalgebra in the non-standard simple root system with two odd simple roots is constructed in [9, 14].

It is well known that the bosonic operators $1,a,a^+$ generate the classical bosonic algebra satisfying the following relations:

$$[a,a^+]=aa^+-a^+a=1,\ [1,a^+]=[1,a]=0. \tag{1.3}$$

By [14, 18], we know that we can use the bosonic operator to realize the Virasoro algebra, the Virasoro superalgebra and the $W(2,2)$ algebra etc. According to [2], the Hopf structure on an algebra is as follows. By a Hopf structure on a algebra A, we mean that A is associated with a triple (Δ,ϵ,S), where the coproduct Δ: $A\to A\otimes A$ is an algebra homomorphism, the counit ϵ: $A\to\mathbb{F}$ is an algebra homomorphism, and the antipode S: $A\to A$ is an anti-homomorphism such that

$$(1\otimes\Delta)\Delta(x)=(\Delta\otimes 1)\Delta(x) \quad \text{(coassociativity)},$$