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The Category Types of Extensions of a Tambara-Yamagami Category

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Abstract. In this paper, we study the group extension of a Tambara-Yamagami category which has Frobenius-Perron dimension $2pq$, where p,q are prime numbers. We prove that there are two possible category types when $p \neq q$, and five possible category types when $p = q$.

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Key words: Fusion category, extension, Tambara-Yamagami category, category type.

1 Introduction

A fusion category $\mathcal{C}$ is a k-linear semisimple rigid finite tensor category such that the unit object $\mathbf{1}$ is simple.

The theory of fusion categories has close connections with Hopf algebras, quantum groups, vertex operator algebras, topological quantum field theory, conformal field theory, and quantum computing [1,6,7,9].

An important result in the theory of fusion categories is that any fusion category is a group extension of its adjoint subcategory [5]. This result naturally prompts us to consider the question: how to classify the extension of a given fusion category? It is impossible to solve this problem at the current state of the art, because it involves at least the classification of representations of semisimple Hopf algebras. But we know little even about the structures of semisimple Hopf algebras except for group algebras. Etingof, Nikshych, and Ostrik, give an explicit description of both the obstructions to the existence of extensions and the data parameterizing them [10], but the method they

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provided is complicated to implement. Therefore, one current feasible and convenient research scheme is to seek for the extensions of some easy fusion categories under a given finite group.

In this paper, we start from a class of Tambara-Yamagami categories, and study the dimension of simple objects and the number of non-isomorphic simple objects in its arbitrary group extensions, so as to obtain their category types. Furthermore, we consider the case when the graded group is a symmetric group, and give a more accurate conclusion in this case.

For the basic theory and notations related to fusion categories in this paper, please refer to [3], but for the completeness of the paper and the convenience of the readers, we will introduce the main concepts and conclusions needed in this paper in the second section.

2 Preliminaries

All fusion categories studied in this paper are defined over an algebraically closed field k of characteristic zero.

2.1 Frobenius-Perron dimension

Let $\mathcal{C}$ be a fusion category, and $K(\mathcal{C})$ be the Grothendieck ring of $\mathcal{C}$. Then the set $\mathrm{Irr}(\mathcal{C})$ consisting of the isomorphism classes of simple objects in $\mathcal{C}$ is the $\mathbb{Z}^+$ basis of $K(\mathcal{C})$. The Frobenius-Perron (FP) dimension $\mathrm{FPdim}(X)$ of the simple object X in $\mathcal{C}$ is defined as the largest eigenvalue of the matrix obtained by the left multiplication of its isomorphism class (still denoted by X) in the basis $\mathrm{Irr}(\mathcal{C})$ of $K(\mathcal{C})$. By the Frobenius-Perron Theorem [4, Theorem 8.1], $\mathrm{FPdim}(X)$ is a positive real number. Moreover, the FP dimension induces a ring homomorphism $\mathrm{FPdim}{:}K(\mathcal{C})\to\mathbb{R}$ [4, Theorem 8.6]. The FP dimension of $\mathcal{C}$ is defined as

$$\mathrm{FPdim}(\mathcal{C})=\sum_{X\in\mathrm{Irr}(\mathcal{C})}\mathrm{FPdim}(X)^2.$$

If $\mathrm{FPdim}(\mathcal{C})$ is an integer, the fusion category $\mathcal{C}$ is said to be weakly integral. If the dimension of every simple object in $\mathcal{C}$ is integral, then the fusion category $\mathcal{C}$ is called integral. The fusion category $\mathcal{C}$ is said to be strictly weakly integral if $\mathcal{C}$ is weakly integral but not integral.

According to [4, Proposition 8.27], if $\mathcal{C}$ is weakly integral, then the FP dimension of a simple object in $\mathcal{C}$ can only be the square root of an integer.

Let $G(\mathcal{C})$ denote the set of isomorphism classes of invertible simple objects of $\mathcal{C}$. Then $G(\mathcal{C})$ is a group with multiplication given by tensor product. The group $G(\mathcal{C})$ acts on the set $\mathrm{Irr}(\mathcal{C})$ by left tensor multiplication. Let $G[X]$ be the stabilizer of $X\in\mathrm{Irr}(\mathcal{C})$ under this action of $G(\mathcal{C})$. Then $g\in G[X]$ if and only if $m(X,g\otimes X)=m(g,X\otimes X^*)=1$, where $m(U,V\otimes W)$ is the multiplicity of U in the decomposition of $V\otimes W$.