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B-Statistical Uniform Integrability and $\mathfrak{L}_p$ Convergence

Mengru Chen[1], Weicai Peng[2] and Zhongzhi Wang[3,*]

[1] *School of General Education, Wanjiang University of Technology, Ma'anshan 243000, China;*
[2] *School of Mathematics & Big Data, Chaohu University, Hefei 238000, China;*
[3] *School of Microelectronics & Data Science, AnHui University of Technology, Ma'anshan 243000, China.*

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Abstract. In this paper, we first give the concept of B-statistical uniform integrability with respect to Cesàro matrix (B-CUI), which is weaker than Cesàro uniform integrability. Then we establish some B-statistical convergence theorems for random sequence under the condition of B-CUI, which generalizes the outcomes of some known results. Finally, for some special kinds of triangular array or pairwise independent sequences of random variables, similar results are also derived without conditions of B-CUI.

AMS subject classifications: 60F25

Key words: B-statistical uniform integral, Cesàro matrix, martingale difference$\mathfrak{L}_p$ convergence.

1 Introduction

The concept of uniform integrability (see Chung (1974)) plays a significant role in the study of probability theory and has always attracted the interest of many researchers. For example, Chandra (1989) extended the concept of uniform integrability called the Cesàro uniform integrability and demonstrated that the $\mathfrak{L}_1$ convergence of the sample mean holds under very general conditions. After that, a series of papers have proved that Cesàro uniform integrability, rather than the stronger uniform integrability condition, can be the correct condition for the law of large numbers (LLNs). Chandra (1992) gave some results on strong law of large numbers (SLLNs) under the Cesàro uniformly integrable condition, and relaxed the condition 'identical distribution and/or independence' in Etemadi's extension (1981) of Kolmogorov SLLNs and in the classical Markov

*Corresponding author. *Email address:* cmr954868550@163.com (Chen M), weicaipeng@126.com (Peng W), zhongzhiw@126.com (Wang Z)

and Cantelli SLLNs. Bose and Chandra (1993) proved that the condition of Cesàro uniform integrability can be used in many cases to prove $\mathfrak{L}_p$ convergence of $n^{-\frac{1}{p}}S_n$. Cabrera (1994) generalized Cesàro uniform integrability to uniform integrability with respect to $\{a_{nk}\}$, and obtained the convergence theorem of weighted sum $\sum_k a_{nk}(X_k - \mathbb{E}X_k)$ under this condition. Chandra and Goswami (2003) introduced a set of new conditions, called Cesàro α-integrability and strong Cesàro α-integrability, under which weak law of large numbers (WLLNs) and SLLNs hold for pairwise independent random variable sequences. Chibisov (2015) discussed the relationship between the WLLNs and the SLLNs for the sum of independent random variables. The mathematical content of these laws and their connections with mathematical statistics problems is compared. The research on uniform integrability of statistical significance has been further carried out recently. Antonini and Ünver (2019) introduced the concept of A-statistical uniform integrability of sequences of random variables, which is weaker than that of uniform integrability, and figured out some characterizations. Cabrera, Rosalsky, Ünver and Volodin (2020) introduced the concept of B-statistical uniform integrability of a sequence of random variables $\{X_k\}$ with respect to $\{a_{nk}\}$. The concept is more general than that of $\{X_k\}$ being uniformly integrable with respect to $\{a_{nk}\}$, and under this condition, a law of large numbers of mean convergence in statistical sense is obtained for pairwise independent random variable sequences.

It is known that the summability theory is particularly important when a sequence of random variables does not converge, because it allows a non-convergent sequence to converge in a more general sense. Motivated by the work of Cabrera, Rosalsky, Ünver and Volodin (2020), we introduce the concept of B-statistical supremum to define B-statistically uniformly integrable with respect to Cesàro matrix (abbreviated by B-CUI), where B is a nonnegative regular summability matrix, which in turn is weaker than the concept of general supremum of mathematical analysis. It can be shown that the concept of B-CUI is more general than Cesàro uniformly integrable. In this paper, we focus on some B-statistical convergences of random sequence under the condition of B-CUI, these convergences are actually a generalization of the work of Bose and Chandra (1993).

The organization of the paper is as follows. In Section 2, we present some definitions and basic properties of B-CUI sequence. Section 3 focuses on the B-statistical convergence of sequences of martingale difference and some special random variables under the condition of B-CUI. B-statistical convergences of random sequence, without conditions of B-CUI, are also discussed.

2 Preliminaries

In this section, we begin by introducing some notations, technical definitions and lemmas which will be used in this paper.

Throughout this paper, the set of all positive integers, the set of all nonnegative integers and the set of all integers will be denoted by $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$, respectively. Let