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Numerical Investigation of the Three-Dimensional Time-Fractional Extended Fisher-Kolmogorov Equation via a Meshless Method

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Abstract. In this paper, we develop an efficient meshless technique for solving numerical solutions of the three-dimensional time-fractional extended Fisher-Kolmogorov (TF-EFK) equation. Firstly, the $L2\text{-}1_\sigma$ formula on a general mesh is used to discretize the Caputo fractional derivative, and then a weighted average technique at two neighboring time levels is adopted to implement the time discretization of the TF-EFK equation. After applying this time discretization, the generalized finite difference method (GFDM) is introduced for the space discretization to solve the fourth-order nonlinear algebra system generated from the TF-EFK equation with an arbitrary domain. Numerical examples are investigated to validate the performance of the proposed meshless GFDM in solving the TF-EFK equation in high dimensions with complex domains.

AMS subject classifications: 33F05, 35G25, 45G05

Key words: Generalized finite difference method, meshless technique, TF-EFK equation, fourth-order nonlinear system, arbitrary domain.

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1 Introduction

The fractional calculus has been widely applied to all related fields of science and engineering [1–15], such as physics, biology, material, mechanics, etc. For instance, the non-integer (between 0 and 1) power of frequency in Cole expression for the membrane reaction was used to fit the experimental data [1]. As pointed out in [1], fractional kinetic equations with the fractional-order derivatives (between 0 and 1) in time govern the ultraslow diffusion, which are called sub-diffusion models. Fractional kinetic equations have gained much success in depicting anomalous diffusion in transport process thorough random environments and non-exponential relaxation patterns [16, 17].

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The extended Fisher-Kolmogorov (EFK) equation is a significant category of nonlinear reaction-diffusion models, frequently encountered in bi-stable systems [18]. The EFK equation can capture essential features of the underlying problems arising in population dynamics [19], chemical kinetics [20], brain tumor dynamics [21], propagation of domain walls in liquid crystals [22], etc. In this paper, we consider a sub-diffusion nonlinear model by introducing a Caputo fractional derivative of order α $(0<\alpha<1)$ into the EFK equation:

$$\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha}+\gamma\Delta^2 u(\mathbf{x},t)-\Delta u(\mathbf{x},t)+G(u(\mathbf{x},t))=f(\mathbf{x},t), \qquad \mathbf{x}\in\Omega,\ t\in(0,T], \tag{1.1}$$

under the given initial condition $u(\mathbf{x},t=0)|_{\mathbf{x}\in\Omega}=u^0(\mathbf{x})$ and boundary conditions

$$u(\mathbf{x},t)|_{\mathbf{x}\in\partial\Omega}=\psi_1(\mathbf{x},t),\ \frac{\partial u(\mathbf{x},t)}{\partial \mathbf{n}}|_{\mathbf{x}\in\partial\Omega}=\psi_2(\mathbf{x},t) \qquad \text{for } t\in(0,T],$$

where the function $u(\mathbf{x},t)$ denotes a physical quantity (e.g., the evolution of a population density); Δ is the Laplace operator; γ is a positive constant; Ω denotes a bounded domain, and $\partial\Omega$ denotes the boundary of Ω; $\mathbf{n}$ denotes the unit normal vector on the boundary; $u^0(\mathbf{x})$, $f(\mathbf{x},t)$, $\psi_1(\mathbf{x},t)$ and $\psi_2(\mathbf{x},t)$ are given functions, and the nonlinear term $G(u(\mathbf{x},t))=u^3(\mathbf{x},t)-u(\mathbf{x},t)$; and in the Caputo fractional derivative definition, $\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha}$ is defined by [1]

$$\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(\mathbf{x},s)}{\partial s}\frac{ds}{(t-s)^\alpha}, \qquad 0<\alpha<1. \tag{1.2}$$

The sub-diffusion nonlinear model (called as the time-fractional EFK equation or the TF-EFK equation) in Eq. (1.1) denotes a fractional form of the EFK equation. When $\alpha=1$, the TF-EFK equation (1.1) reduces to the integer-order EFK equation given in [18]. It is well known that the study of the solution of the EFK-type model with the initial and boundary conditions (IBCs) tends to capture the essential features of the dynamics in complex systems. However, it is difficult to work out the analytical solution of the EFK-type model with IBCs, since it is a class of nonlinear model. Accordingly, the numerical study of the EFK-type model with IBCs is critical. For the integer-order EFK model, there exist many research studies on numerical methods, such as the finite difference method [23–27], the finite element method [28–31], the collocation method [32], etc. For the fractional EFK-type model, based on the literature review, the study of its numerical methods is relatively limited. Shamseldeen et al. [33] used the optimal homotopy analysis method to compute the approximate solution of a 1D space-time fractional EFK equation. Hosseininia et al. [34] proposed a numerical method for solving the 2D variable-order fractional EFK equation on the regular geometries (i.e., rectangular) based on the orthonormal shifted discrete Legendre polynomials and the collocation method.

It is should be noted that when solving partial differential equations (PDEs) in complex areas (e.g., irregular areas in high-dimension), the numerical simulation of boundary domain based on the mesh-dependent numerical methods, like the finite difference method, is tedious in general.