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On the Computational Problems of Upper Convex Densities for Self-Similar Sets with the Open Set Condition

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Abstract. Let E be a self-similar set satisfying the open set condition. Zhou and Feng posed an open problem in 2004 as follows: let $x\in E$, under what conditions is there a set U_x containing x with $|U_x|>0$ such that $\overline{D}_C^s(E,x)=\frac{\mathcal{H}^s(E\cap U_x)}{|U_x|^s}$? The aim of this paper is to present a solution of this problem. Under the assumption that there exists a nonempty convex open set V containing E and satisfying the requirement of the open set condition, it is proved that if $x\in E$ and the upper convex density of E at x equals 1, then there exists a convex set U_x containing x with $|U_x|>0$ such that $\overline{D}_C^s(E,x)=\frac{\mathcal{H}^s(E\cap U_x)}{|U_x|^s}$. Finally, as an application of this result, an equivalent condition for $E_0=E$ is given, where $E_0=\{x\in E|\overline{D}_C^s(E,x)=1\}$.

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Key words: Hausdorff measure, self-similar set, open set condition, upper convex density.

1 Introduction and basic notations

Hausdorff measure and Hausdorff dimension are two basic notions of fractals, so how to calculate or estimate the Hausdorff measures and Hausdorff dimensions of fractal sets becomes a compelling problem. In general, it is very hard to calculate or estimate Hausdorff measures and Hausdorff dimensions of fractal sets, especially to calculate their Hausdorff measures ([5, 6, 9]). Up to now, there are few calculations on Hausdorff measures of fractal sets whose Hausdorff dimensions are larger than 1 ([1–3]). Why is it so difficult to calculate the Hausdorff measures of fractals? As Zhou and Feng said in [10], "the reason is neither computational trickiness nor computational capacity, but a lack of full understanding of the essence of Hausdorff measure". For the sake of looking into

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the intrinsic properties of fractals, Zhou and Feng [10] gave the notions of best covering and best shape and posed eight open problems and six conjectures on the calculation of Hausdorff measures of self-similar sets. Among them is an open problem as follows (see Problem 3 of [10]).

Problem 1.1 (The finite realization problem [10])**.** Let $E\subset\mathbb{R}^n$ be a self-similar set satisfying the open set condition, $x\in E$ and s the Hausdorff dimension of E. Under what conditions is there a set U_x containing x with $|U_x|>0$ such that $\overline{D}_C^s(E,x)=\frac{\mathcal{H}^s(E\cap U_x)}{|U_x|^s}$, that is, U_x realizes the calculation of the upper convex density of E at x?

The upper convex density of E at $x\in E$ is defined as follows:

$$\overline{D}_C^s(E,x)=\limsup_{\delta\to 0,0<|U_x|\leqslant\delta}\left\{\frac{\mathcal{H}^s(E\cap U_x)}{|U_x|^s}:x\in U_x,U_x \text{ is convex}\right\}. \tag{1.1}$$

This open problem is very important because it is closely related to the calculation of Hausdorff measures. In fact, let E be a self-similar set and $x\in E$ at which the upper convex density of E equals 1, if there is a set realizing the calculation of the upper convex density of E at x, namely, there exists a set U with $|U|>0$ such that $\mathcal{H}^s(E\cap U)=|U|^s$, then for each $k\geq 1$, by counting how many k-contracting-copies are contained in U but not contained in all $k-1$-contracting-copies in U, we can theoretically calculate the exact value of the Hausdorff measure of E. But it is just a theoretical calculating method, the calculation of the exact values of Hausdorff measures depends also on how to determine the diameter and the shape, even the position of U in E.

In this work, under the assumption that there exists a nonempty convex open set V which contains E and fulfills the requirements of the open set condition, we prove that for the point $x\in E$ with the upper convex density equalling 1, there exists a convex set U_x containing x with $|U_x|>0$ such that U_x realizes the calculation of the upper convex density of E at x. As consequence, we partly answer the open Problem 1.1. Finally, as an application, we offer an equivalent condition for $E_0=E$.

For some known definitions such as Hausdorff measure, Hausdorff dimension and some known results, we refer to [1–4].

Let $A\subset\mathbb{R}^n$ be nonempty, $\delta>0$. Set $V(A,\delta)=\{x\in\mathbb{R}^n : d(A,x)<\delta\}$ and denote $\mathcal{C}$ the set consisting of all compact subsets of $\mathbb{R}^n$. Suppose $A,B\in\mathcal{C}$, recall the definition of Hausdorff metric as follows:

$$\rho(A,B)=\inf\left\{\delta:\overline{V(A,\delta)}\supset B,\overline{V(B,\delta)}\supset A\right\}.$$

We endow $\mathcal{C}$ with the metric ρ, then $\mathcal{C}$ is a complete metric space.

Let $S=\{1,\cdots,m\}$ $(m\geqslant 2)$. The one-sided symbolic space generated by S is denoted as $\Sigma_m=\{i=(i_1i_2\cdots)|\ i_j\in S,j\geqslant 1\}$. For each $k\geqslant 1$, denote by J_k the set of k-sequences in the form of $(j_1,\cdots,j_k)$, where $1\leqslant j_1,\cdots,j_k\leqslant m$, $k\geqslant 1$. Put

$$E_{i_1\cdots i_k}=S_{i_1}\circ\cdots\circ S_{i_k}(E),\qquad \forall i=(i_1\cdots i_k)\in J_k,$$