J. Math. Study
doi: 10.4208/jms.v57n4.24.08

Vol. **57**, No. 4, pp. 509-527
December 2024

Normalized Solutions for a Kirchhoff Equation with Potential in $\mathbb{R}^3$

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Received December 7, 2023; Accepted March 14, 2024;
Published online December 24, 2024.

Abstract. In this paper, for given mass $c>0$, we study the existence of normalized solutions to the following nonlinear Kirchhoff equation

$$\begin{cases} \left(a+b\int_{\mathbb{R}^3}[|\nabla u|^2+V(x)u^2]dx\right)[-\Delta u+V(x)u]=\lambda u+\mu|u|^{q-2}u+|u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3}|u|^2dx=c^2, \end{cases}$$

where $a>0, b>0, \lambda\in\mathbb{R}$, $5<q<p<6$, $\mu>0$ and V is a continuous non-positive function vanishing at infinity. Under some mild assumptions on V, we prove the existence of a mountain pass normalized solution via the minimax principle.

AMS subject classifications: 35C15, 35Q51

Key words: Kirchhoff equation, normalized solutions, minimax principle.

1 Introduction

In this paper, we study the existence of normalized solutions to the following nonlinear Kirchhoff equation with potential

$$\left(a+b\int_{\mathbb{R}^3}[|\nabla u|^2+V(x)u^2]dx\right)[-\Delta u+V(x)u]=\lambda u+\mu|u|^{q-2}u+|u|^{p-2}u, \text{ in } \mathbb{R}^3, \tag{1.1}$$

where $a>0$, $b>0$, $\lambda\in\mathbb{R}$, $\mu>0$ and $5<q<p<6$.

Problem like Eq. (1.1) was proposed by Kirchhoff [1] in 1883. In (1.1), a is related to the intrinsic properties of strings, b denotes initial tension. The equation is not a pointwise identity but is nonlocal because of $\int_{\mathbb{R}^3}|\nabla u|^2dx$. This phenomenon has been widely used in biological systems and has inspired many scholars to study this kind of problem.

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At present, there are two different ways to study Eq. (1.1). One is to fix $\lambda\in\mathbb{R}$ to find the solutions u of Eq. (1.1). In this situation, solutions are critical points of the following action functional $F:H^1(\mathbb{R}^3)\to\mathbb{R}$, defined by

$$F(u)=\frac{a}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+V(x)|u|^2)dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}(|\nabla u|^2+V(x)|u|^2)dx\right)^2-\frac{\lambda}{2}\int_{\mathbb{R}^3}|u|^2dx$$
$$-\frac{\mu}{q}\int_{\mathbb{R}^3}|u|^q dx-\frac{1}{p}\int_{\mathbb{R}^3}|u|^p dx.$$

And the other one is to consider the case $\lambda\in\mathbb{R}$ is unknown. Alternatively, one can search for solutions to Eq. (1.1) having prescribed mass

$$\int_{\mathbb{R}^3}|u|^2dx=c^2. \tag{1.2}$$

The solution of Eq. (1.1) satisfies the prescribed mass constraint (1.2), which is called the fixed mass problem, and called the normalized solution.

If $V(x)=0$, they studied the existence of normalized solutions of Eq. (1.1) in Li [6] and Kong [10]. When $V(x)\neq 0$, they studied the problem similar to Eq. (1.1) in [5,15-16]. In Li [6], the author discussed the existence of solutions for a class of Kirchhoff equations, extending the results of Soave [3]. They developed a perturbed Pohozaev constrained method to tackle the difficulties which arising from the non-local term in the Kirchhoff equation. They performed delicate energy estimates under L^2 constraint to restore compactness at the critical Sobolev level.

This paper is inspired by Kang [5], which studied the normalized solutions for the nonlinear Schrödinger equation with potential and combined nonlinearities, we are interested in the existence of normalized solution of Eq. (1.1) and our result seems to extend the results of [5] and [6]. We use the minimax method and some results in [6] to study the existence of a mountain pass normalized solution. Throughout this paper we will make the following assumptions on V:

(V_1) $\lim_{|x|\to\infty}V(x)=\sup_{x\in\mathbb{R}^3}V(x)=0$, $V(x)=V(|x|)$, there exists $0<\delta_1<\frac{1}{9}$, such that for any $u\in H_r^1(\mathbb{R}^3)$,
$$\left|\int_{\mathbb{R}^3}V(x)|u|^2dx\right|\leq\delta_1|\nabla u|_2^2.$$

(V_2) $\nabla V(x)$ exists for a.e. $x\in\mathbb{R}^3$, define
$$W(x):=\frac{1}{2}\langle\nabla V(x),x\rangle.$$
There exists $0<\delta_2<\min\{\frac{(3q-6)(1-\delta_1)}{8}-1,\frac{1}{9}\}$, such that for any $u\in H_r^1(\mathbb{R}^3)$
$$\left|\int_{\mathbb{R}^3}W(x)|u|^2dx\right|\leq\delta_2|\nabla u|_2^2.$$