J. Math. Study
doi: 10.4208/jms.v57n4.24.09

Vol. **57**, No. 4, pp. 528-538
December 2024

# Endpoint Estimates for an Oscillatory Multiplier Associated with Wave Equations on the Torus

Ziyao Liu[1,*] and Dashan Fan[2]

[1] *Department of Mathematical Science, Zhejiang Normal University, Jinhua 321004, China;*
[2] *Department of Mathematical Science, University of Wisconsin-Milwaukee, Milwaukee WI 53201, USA.*

Received December 15, 2023; Accepted September 5, 2024;
Published online December 24, 2024.

**Abstract.** In this paper, we establish the endpoint estimate for an oscillatory multiplier associated with wave equations on the torus, which extends the results of Fan and Sun. In addition, we obtain a more general result for sublinear operators on compact measure spaces.

**AMS subject classifications**: 42B15, 42B25, 42B99

**Key words**: Oscillatory multiplier, wave equation, compact measure spaces.

## 1 Introduction

In this article, we concern the Cauchy problem of the wave equation

$$\begin{aligned}&\partial_{tt}u(x,t)-\Delta u(x,t)=0, \qquad (x,t)\in\mathbb{R}^n\times\mathbb{R},\\ &u(x,0)=0, \qquad \partial_t u(x,0)=f(x),\end{aligned}$$

where $\Delta$ denotes the Laplacian on $\mathbb{R}^n$. The solution $u$ is formally given by

$$u(x,t)=\left(\frac{\sin(t\sqrt{|\Delta|})}{\sqrt{|\Delta|}}f\right)(x).$$

To study the behavior of the solution, we need to write the operator $\frac{\sin(t\sqrt{|\Delta|})}{\sqrt{|\Delta|}}$ as a sum of its high frequency part and low frequency part. For this reason, we pick two radial

*Corresponding author. *Email address:* `zy.liu@zjnu.edu.cn` (Liu Z), `fandashan2@zjnu.edu.cn` (Fan D)

functions $\Phi,\Psi \in C^\infty(\mathbb{R}^n)$ satisfying that $\Psi(\lambda)\equiv 1$ on the set $\{\lambda : |\lambda| > 5/3\}$ and is supported on the set $\{\lambda : |\lambda| > 3/5\}$, and

$$\Phi(\lambda) = 1 - \Psi(\lambda).$$

Then we write

$$\frac{\sin(t\sqrt{|\Delta|})}{\sqrt{|\Delta|}} = \frac{\sin(t\sqrt{|\Delta|})}{\sqrt{|\Delta|}}\Phi\left(t\sqrt{|\Delta|}\right) + \frac{\sin(t\sqrt{|\Delta|})}{\sqrt{|\Delta|}}\Psi\left(t\sqrt{|\Delta|}\right).$$

It is easy to check that

$$\frac{\sin(t\sqrt{|\Delta|})}{\sqrt{|\Delta|}}\Phi\left(t\sqrt{|\Delta|}\right)$$

is bounded on $L^p(\mathbb{R}^n)$ for any $p \geq 1$. Thus to study the $L^p(\mathbb{R}^n)$ boundedness of $\frac{\sin(t\sqrt{|\Delta|})}{\sqrt{|\Delta|}}$, now it suffices to study the $L^p(\mathbb{R}^n)$ boundedness of $\frac{\sin(\sqrt{|\Delta|})}{\sqrt{|\Delta|}}\Psi\left(\sqrt{|\Delta|}\right)$, where we may fix $t=1$ without loss of generality ([3,7,8,10]).

By checking the Fourier transform, it is not difficult to see that the multiplier of $\frac{\sin(\sqrt{|\Delta|})}{\sqrt{|\Delta|}}\Psi\left(\sqrt{|\Delta|}\right)$ is

$$\frac{\sin|\xi|}{|\xi|}\Psi(|\xi|).$$

Thus, to obtain $L^p_\beta \to L^p$ boundedness, we naturally consider a general oscillatory integral operator on $\mathbb{R}^n$ defined by

$$S_{\alpha,m}(f) = K_{\alpha,m} * f,$$

where $K_{\alpha,m}$ is a distribution kernel whose Fourier transform is

$$\widehat{K_{\alpha,m}}(\xi) = e^{ci|\xi|}|\xi|^{-\alpha}\Omega(\xi')\Psi(|\xi|)(\log|\xi|)^m, \qquad \alpha,m \geq 0,$$

where $c=1$ or $-1$.

Here, $\Omega(\xi') = \Omega(\frac{\xi}{|\xi|})$ is a $C^\infty$ non-negative and non-zero function on the unit sphere $S^{n-1}$ if $n \geq 2$ and $\Omega(\xi')=1$ if $n=1$. By this definition, $u(x,1)$ is a special case of the operator $S_{\alpha,m}(f)$.

On the n-torus $T^n$, we similarly define the operator

$$\begin{aligned}\widetilde{S}_{\alpha,m}(g)(x) &= \widetilde{K}_{\alpha,m} * g(x)\\ &= \sum_{j\in\mathbb{Z}^n} c_j \widehat{K_{\alpha,m}}(j) e^{2\pi i j\cdot x},\end{aligned}$$

where $g \in L^1(T^n)$ and

$$g(x) \sim \sum_{j\in\mathbb{Z}^n} c_j e^{2\pi i j\cdot x}.$$