

Ricci Curvature and Fundamental Groups of Effective Regular Sets

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In honor of Professor Xiaochun Rong on his seventieth birthday

Abstract. For a Gromov-Hausdorff convergent sequence of closed manifolds $M_i^n \xrightarrow{GH} X$ with $\text{Ric} \geq -(n-1)$, $\text{diam}(M_i) \leq D$, and $\text{vol}(M_i) \geq v > 0$, we study the relation between $\pi_1(M_i)$ and X . It was known before that there is a surjective homomorphism $\phi_i : \pi_1(M_i) \rightarrow \pi_1(X)$ by the work of Pan-Wei. In this paper, we construct a surjective homomorphism from the interior of the effective regular set in X back to M_i , that is, $\psi_i : \pi_1(\mathcal{R}_{\epsilon, \delta}^\circ) \rightarrow \pi_1(M_i)$. These surjective homomorphisms ϕ_i and ψ_i are natural in the sense that their composition $\phi_i \circ \psi_i$ is exactly the homomorphism induced by the inclusion map $\mathcal{R}_{\epsilon, \delta}^\circ \hookrightarrow X$.

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1 Introduction

For a Gromov-Hausdorff convergent sequence $M_i \xrightarrow{GH} X$ with curvature bounds, it is crucial to understand the relationship between M_i and X . For example, when M_i are closed n -manifolds with

$$\text{sec} \geq -1, \quad \text{diam}(M_i) \leq D, \quad \text{vol}(M_i) \geq v > 0,$$

Perelman proved that M_i is homeomorphic to X for all i large [14]. For the context of this paper, let us consider a convergent sequence of closed n -manifolds $M_i \xrightarrow{GH} X$ with Ricci curvature lower bounds

$$\text{Ric} \geq -(n-1), \quad \text{diam}(M_i) \leq D, \quad \text{vol}(M_i) \geq v > 0 \tag{1.1}$$

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Under this weaker condition, one cannot expect X to be homeomorphic to M_i . By the work of Wei and the author [13], the limit space X is semi-locally simply connected. This was later generalized to the collapsing case by Wang [16]. As a consequence, there is a forward surjective homomorphism from $\pi_1(M_i)$ to $\pi_1(X)$.

Theorem 1.1 ([13]). *Let M_i be a sequence of closed n -manifolds with (1.1) and Gromov-Hausdorff converging to a limit space X . Let $x_i \in M_i$ be a sequence of points converging to $x \in X$. Then for all i large, there is a surjective homomorphism*

$$\phi_i: \pi_1(M_i, x_i) \rightarrow \pi_1(X, x).$$

For an element $[\sigma_i] \in \pi_1(M_i, x_i)$ represented by a loop σ_i based at x_i , its image under this forward homomorphism ϕ_i is constructed by drawing a loop σ in X that is sufficiently close to σ_i , see [13, 15]. While ϕ_i is surjective, in general it is not injective even under the noncollapsing condition. In fact, there could be shorter and shorter non-contractible loops at x_i with length tending to 0, then by construction ϕ_i sends them to identity. We will review an example by Otsu [11] in Section 3 regarding this.

From Theorem 1.1, because ϕ_i may have a kernel, it appears that some elements in $\pi_1(M_i)$ are lost in the limit X . As the main result of this paper, we show that all elements in $\pi_1(M_i)$ are still retained in X ; more specifically, in the effective regular set $\mathcal{R}_{\epsilon, \delta}$ of X . In fact, we will construct a backward surjective homomorphism from $\pi_1(\mathcal{R}_{\epsilon, \delta}^\circ, x)$ to $\pi_1(M_i, x_i)$, where $\mathcal{R}_{\epsilon, \delta}^\circ$ is the interior of $\mathcal{R}_{\epsilon, \delta}$ and $x \in X$ is a regular point. By the regularity theory developed by Cheeger-Colding [4], $\mathcal{R}_{\epsilon, \delta}^\circ$ is a connected topological manifold of dimension n for all $0 < \epsilon \leq \epsilon(n)$ and $\delta > 0$.

Theorem 1.2. *Let*

$$(M_i, x_i) \xrightarrow{GH} (X, x)$$

be a convergent sequence of closed n -manifolds with (1.1), where x is a regular point. Then

- (1) *for any $0 < \epsilon < \epsilon(n)$ and sufficiently small $0 < \delta < \delta(\epsilon, x)$, there is a surjective homomorphism*

$$\psi_i^\delta: \pi_1(\mathcal{R}_{\epsilon, \delta}^\circ, x) \rightarrow \pi_1(M_i, x_i)$$

for all i large;

- (2) *the composition of ψ_i^δ and ϕ_i in Theorem 1.1*

$$\phi_i \circ \psi_i^\delta: \pi_1(\mathcal{R}_{\epsilon, \delta}^\circ, x) \rightarrow \pi_1(X, x)$$

is exactly the homomorphism ι_ induced by the inclusion map $\iota: \mathcal{R}_{\epsilon, \delta}^\circ \hookrightarrow X$.*

The construction of this backward homomorphism ψ_i is natural and similar to that of ϕ_i : namely, by drawing nearby loops. The surjectivity of ψ_i requires a complete different and more involved argument than that of ϕ_i . We remark that ψ_i is not injective in general. In