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Ricci Curvature and Fundamental Groups of Effective Regular Sets

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In honor of Professor Xiaochun Rong on his seventieth birthday

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Abstract. For a Gromov-Hausdorff convergent sequence of closed manifolds $M_i^n \xrightarrow{GH} X$ with $\mathrm{Ric} \geq -(n-1)$, $\mathrm{diam}(M_i) \leq D$, and $\mathrm{vol}(M_i) \geq v > 0$, we study the relation between $\pi_1(M_i)$ and X. It was known before that there is a surjective homomorphism $\phi_i : \pi_1(M_i) \to \pi_1(X)$ by the work of Pan-Wei. In this paper, we construct a surjective homomorphism from the interior of the effective regular set in X back to M_i, that is, $\psi_i : \pi_1(\mathcal{R}^{\circ}_{\epsilon,\delta}) \to \pi_1(M_i)$. These surjective homomorphisms ϕ_i and ψ_i are natural in the sense that their composition $\phi_i \circ \psi_i$ is exactly the homomorphism induced by the inclusion map $\mathcal{R}^{\circ}_{\epsilon,\delta} \hookrightarrow X$.

AMS subject classifications: 53C21, 53C23

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1 Introduction

For a Gromov-Hausdorff convergent sequence $M_i \xrightarrow{GH} X$ with curvature bounds, it is crucial to understand the relationship between M_i and X. For example, when M_i are closed n-manifolds with

$$\sec \geq -1, \qquad \mathrm{diam}(M_i) \leq D, \qquad \mathrm{vol}(M_i) \geq v > 0,$$

Perelman proved that M_i is homeomorphic to X for all i large [14]. For the context of this paper, let us consider a convergent sequence of closed n-manifolds $M_i \xrightarrow{GH} X$ with Ricci curvature lower bounds

$$\mathrm{Ric} \geq -(n-1), \qquad \mathrm{diam}(M_i) \leq D, \qquad \mathrm{vol}(M_i) \geq v > 0 \tag{1.1}$$

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Under this weaker condition, one cannot expect X to be homeomorphic to M_i. By the work of Wei and the author [13], the limit space X is semi-locally simply connected. This was later generalized to the collapsing case by Wang [16]. As a consequence, there is a forward surjective homomorphism from $\pi_1(M_i)$ to $\pi_1(X)$.

Theorem 1.1 ([13]). *Let M_i be a sequence of closed n-manifolds with* (1.1) *and Gromov-Hausdorff converging to a limit space X. Let $x_i \in M_i$ be a sequence of points converging to $x \in X$. Then for all i large, there is a surjective homomorphism*

$$\phi_i : \pi_1(M_i, x_i) \to \pi_1(X, x).$$

For an element $[\sigma_i] \in \pi_1(M_i, x_i)$ represented by a loop σ_i based at x_i, its image under this forward homomorphism ϕ_i is constructed by drawing a loop σ in X that is sufficiently close to σ_i, see [13, 15]. While ϕ_i is surjective, in general it is not injective even under the noncollapsing condition. In fact, there could be shorter and shorter non-contractible loops at x_i with length tending to 0, then by construction ϕ_i sends them to identity. We will review an example by Otsu [11] in Section 3 regarding this.

From Theorem 1.1, because ϕ_i may have a kernel, it appears that some elements in $\pi_1(M_i)$ are lost in the limit X. As the main result of this paper, we show that all elements in $\pi_1(M_i)$ are still retained in X; more specifically, in the effective regular set $\mathcal{R}_{\epsilon,\delta}$ of X. In fact, we will construct a backward surjective homomorphism from $\pi_1(\mathcal{R}^\circ_{\epsilon,\delta}, x)$ to $\pi_1(M_i, x_i)$, where $\mathcal{R}^\circ_{\epsilon,\delta}$ is the interior of $\mathcal{R}_{\epsilon,\delta}$ and $x \in X$ is a regular point. By the regularity theory developed by Cheeger-Colding [4], $\mathcal{R}^\circ_{\epsilon,\delta}$ is a connected topological manifold of dimension n for all $0 < \epsilon \leq \epsilon(n)$ and $\delta > 0$.

Theorem 1.2. *Let*

$$(M_i, x_i) \xrightarrow{GH} (X, x)$$

be a convergent sequence of closed n-manifolds with (1.1), *where x is a regular point. Then*

(1) *for any $0<\epsilon<\epsilon(n)$ and sufficiently small $0<\delta<\delta(\epsilon,x)$, there is a surjective homomorphism*

$$\psi_i^\delta : \pi_1(\mathcal{R}^\circ_{\epsilon,\delta}, x) \to \pi_1(M_i, x_i)$$

for all i large;

(2) *the composition of ψ_i^δ and ϕ_i in Theorem* 1.1

$$\phi_i \circ \psi_i^\delta : \pi_1(\mathcal{R}^\circ_{\epsilon,\delta}, x) \to \pi_1(X, x)$$

is exactly the homomorphism $\iota_\star$ induced by the inclusion map $\iota : \mathcal{R}^\circ_{\epsilon,\delta} \hookrightarrow X$.

The construction of this backward homomorphism ψ_i is natural and similar to that of ϕ_i: namely, by drawing nearby loops. The surjectivity of ψ_i requires a complete different and more involved argument than that of ϕ_i. We remark that ψ_i is not injective in general. In fact, we will review an example by Anderson [1] in Section 3; in this example, both M_i and X are simply connected but $\pi_1(\mathcal{R}^\circ_{\epsilon,\delta})$ is isomorphic to $\mathbb{Z}_2$.

As an application of Theorem 1.1, we show that if the inclusion map $\mathcal{R}^\circ_\epsilon \hookrightarrow X$ induces an injective homomorphism $\iota_\star:\pi_1(\mathcal{R}^\circ_\epsilon,x)\to\pi_1(X,x)$, then $\pi_1(M_i)$ is isomorphic to $\pi_1(X)$ for all i large. Note that we are considering the ϵ-regular set in this statement; in other words, the involvement of δ is dropped.

Theorem 1.3. *Let*

$$(M_i,x_i)\xrightarrow{GH}(X,x)$$

be a convergent sequence of closed n-manifolds with (1.1), where x is a regular point. Suppose that for some $0<\epsilon<\epsilon(n)$, the induced homomorphism

$$\iota_\star:\pi_1(\mathcal{R}^\circ_\epsilon,x)\to\pi_1(X,x)$$

is injective, then $\pi_1(M_i)$ is isomorphic to $\pi_1(X)$ for all i large. In particular, if $\mathcal{R}^\circ_\epsilon$ is simply connected, then so is M_i.

The work in this paper is motivated by the π_1-stability problem:

Question 1.1. Given $n,D,v>0$, is there a positive constant $\epsilon(n,D,v)>0$ such that if two closed n-manifolds M_1 and M_2 satisfy (1.1) and $d_{GH}(M_1,M_2)\le\epsilon$, then are $\pi_1(M_1)$ and $\pi_1(M_2)$ isomorphic?

As a comparison, if one replaces Ricci curvature in (1.1) by a sectional curvature lower bound $\sec\ge -1$, then M_1 and M_2 are homeomorphic when they are Gromov-Hausdorff close; see the works by Grove-Petersen-Wu [10] and Perelman [14].

Question 1.1 is a stronger version of the celebrated finiteness result by Anderson [1] below. In fact, if Question 1.1 has an affirmative answer, then finiteness would easily follow by a standard contradicting argument.

Theorem 1.4 ([1])**.** *Given $n,D,v>0$, there are finitely many isomorphism classes of fundamental groups among closed n-manifolds with (1.1).*

To resolve Question 1.1, it is equivalent to answer:

Question 1.2. For a convergent sequence of closed n-manifolds $M_i\xrightarrow{GH}X$ with (1.1), is it possible to determine $\pi_1(M_i)$ solely from X?

Theorems 1.2 and 1.3 provide partial answers to Question 1.2.

Remark 1.1. Let us mention other related results regarding Questions 1.1 and 1.2.

(1) When X satisfies a local half-volume lower bound, we have a positive answer; see [13, Section 3].

(2) If one considers the equivariant Gromov-Hausdorff convergence of the Riemannian universal covers, then it holds that $\pi_1(M_i,p_i)$ is isometric to the limit group for all i large ([12, Section 2.3] for details). However, because a subsequence was chosen to derive equivariant convergence, this result does not provide direct answers to Question 1.1.

The proof of Theorem 1.2 consists of two steps. The first step is to construct the map ψ_i^δ and show that it is well-defined for small δ. The second step is to show its surjectivity. The proofs relies on several ingredients. The first one is the regularity theory of non-collapsing Ricci limit spaces developed by Cheeger-Colding [3–5,7]. The second ingredient is the equivariant convergence under Ricci and volume lower bounds; in particular, we utilize some of the results by Pan-Rong [12] and Chen-Rong-Xu [6]. Lastly, we use some of the methods in Pan-Wei's work [13] on loops and homotopies under Gromov-Hausdorff convergence; these techniques can be traced back to the work of Borsuk [2] and Tuschman [15].

2 Preliminaries

2.1 Regularity theory of noncollapsing Ricci limit spaces

Throughout the paper, we always use $\Psi(\epsilon|n)$ to represent some nonnegative function depending on ϵ and n with

$$\lim_{\epsilon\to 0}\Psi(\epsilon|n)=0.$$

We may use the same symbol $\Psi(\epsilon|n)$ though dependence on ϵ or n may be different.

Given $n\in\mathbb{N}$, $\kappa\geq 0$, and $v>0$, we denote $\mathcal{M}(n,-\kappa,v)$ the set of all pointed Ricci limit spaces (X,x) coming from some GH convergent sequence of complete n-manifolds (M_i,p_i) with

$$\mathrm{Ric}\geq -(n-1)\kappa,\qquad \mathrm{vol}(B_1(p_i))\geq v>0. \tag{2.1}$$

The regularity theory about these noncollapsing Ricci limit spaces are mainly developed by Cheeger, Colding, and Naber. Below, we review some of the results that will be used later. The main references are [3,5].

Definition 2.1 ([3,5]). *Let $\epsilon,\delta>0$. For a Ricci limit space $X\in\mathcal{M}(n,-1,v)$, we define (ϵ,δ)-regular set, ϵ-regular set, regular set, and singular set of X as below.*

$$\begin{aligned}
&\mathcal{R}_{\epsilon,\delta}=\{x\in X \mid d_{GH}(B_r(x),B_r^n(0))\leq \epsilon r \textit{ for all } 0<r\leq\delta\},\\
&\mathcal{R}_\epsilon=\bigcup_{\delta>0}\mathcal{R}_{\epsilon,\delta},\\
&\mathcal{R}=\bigcap_{\epsilon>0}\mathcal{R}_\epsilon=\bigcap_{\epsilon>0}\bigcup_{\delta>0}\mathcal{R}_{\epsilon,\delta}.\\
&\mathcal{S}=X-\mathcal{R}.
\end{aligned}$$

Theorem 2.1 ([5,7]). *Let $(M_i^n,p_i)\xrightarrow{GH}(X,x)$ be a convergent sequence with* (2.1). *Then for all $r>0$, we have volume convergence*

$$\mathrm{vol}(B_r(p_i))\to\mathcal{H}^n(B_r(x))$$

as $i\to\infty$, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure on X.

Theorem 2.2 ([5,7]). *Let $X\in\mathcal{M}(n,-\delta,v)$ and $x\in X$.*

(1) *If*

$$d_{GH}(B_1(x),B_1^n(0))\le\delta,$$

then

$$\mathcal{H}^n(B_1(x))\ge(1-\Psi(\delta|n))\mathrm{vol}(B_1^n(0)).$$

(2) *If*

$$\mathcal{H}^n(B_1(x))\ge(1-\delta)\mathrm{vol}(B_1^n(0)),$$

then

$$d_{GH}(B_1(x),B_1^n(0))\le\Psi(\delta|n).$$

The following facts follow from Theorems 2.1 and 2.2.

Lemma 2.1. *Let $X\in\mathcal{M}(n,-1,v)$.*
(1) *Given $\epsilon,\delta>0$, there are $\epsilon'=\Psi(\epsilon|n)$ and $\delta'=\delta/3$ such that $\mathcal{R}_{\epsilon,\delta}\subseteq\mathcal{R}^\circ_{\epsilon',\delta'}$.*
(2) *Let A be a compact subset of $\mathcal{R}$. Then for any $\epsilon>0$, there is $\delta>0$ such that $A\subseteq\mathcal{R}_{\epsilon,\delta}$.*

Proof. We include the proof here for readers' convenience.

(1) Let $x\in\mathcal{R}_{\epsilon,\delta}$. By definition, this means

$$d_{GH}(B_r(x),B_r^n(0))\le\epsilon r$$

for all $0<r\le\delta$. By Theorem 2.2 and Bishop-Gromov relative volume comparison,

$$\mathcal{H}^n(B_s(y))\ge(1-\Psi(\epsilon|n))\mathrm{vol}(B_s^n(0))$$

holds for all $y\in B_{\delta/3}(x)$ and all $0<s\le\delta/3$. Applying Theorem 2.2 (2), we see that

$$d_{GH}(B_s(y),B_s^n(0))\le\Psi(\epsilon|n)s,$$

that is, $y\in\mathcal{R}_{\Psi(\epsilon|n),\delta/3}$ for all $y\in B_{\delta/3}(x)$. Therefore, $x\in\mathcal{R}^\circ_{\epsilon',\delta'}$, where $\epsilon'=\Psi(\epsilon|n)$ and $\delta'=\delta/3$.

(2) Let $\epsilon>0$. For each $x\in A$, we pick $\delta(x)>0$ as the largest δ so that

$$d_{GH}(B_r(x),B_r^n(0))\le\epsilon r$$

holds for all $0<r\le\delta$. It suffices to show that $\delta(x)$ has a uniform positive lower bound for all $x\in A$. We argue by contradiction. Suppose that there is a sequence $x_i\in A$ with $\delta(x_i)\to 0$. Then by compactness of A, x_i subconverges to some $y\in A$, which is also regular. Therefore, for $\epsilon'>0$, which will be determined later, there is $\delta_0=\delta_0(\epsilon',y)>0$ such that $y\in\mathcal{R}_{\epsilon',\delta_0}$. Thus it follows from Theorem 2.2(1) that

$$\mathcal{H}^n(B_{\delta_0}(y))\ge(1-\Psi(\epsilon'|n))\mathrm{vol}(B^n_{\delta_0}(0)).$$

By volume convergence,

$$\mathcal{H}^n(B_{\delta_0}(x_i))\ge(1-2\Psi(\epsilon'|n))\mathrm{vol}(B^n_{\delta_0}(0))$$

for i large, thus

$$d_{GH}(B_r(x_i),B_r^n(0))\le\Psi'(\epsilon'|n)r$$

for all $0<r\le\delta_0$. Now we choose $\epsilon'>0$ so that $\Psi'(\epsilon'|n)\le\epsilon$, then $x_i\in\mathcal{R}_{\epsilon,\delta_0}$ for all i large. A contradiction to $\delta(x_i)\to 0$. This completes the proof. □

Theorem 2.3 ([5]). *Let $X\in\mathcal{M}(n,-1,v)$. Then its singular set $\mathcal{S}$ has Hausdorff dimension at most $n-2$.*

Theorem 2.4 ([5]). *Let $X\in\mathcal{M}(n,-1,v)$ and let A be a closed subset of X with $\mathcal{H}^{n-1}(A)=0$. Then $X-A$ is path connected. Moreover, given any $\delta>0$ and any pair of points $x,y\in X-A$, a path σ in $X-A$ between x,y can be chosen that*

$$\mathrm{length}(\sigma)\le(1+\delta)d(x,y).$$

Theorem 2.5 ([5]). *Given dimension n, there is a constant $\epsilon_0(n)>0$ such that the following holds for all $0<\epsilon\le\epsilon_0(n)$. Let $X\in\mathcal{M}(n,-1,v)$ and $x\in X$ such that*

$$d_{GH}(B_\delta(x),B_\delta^n(0))\le\epsilon\delta,$$

where $\delta>0$. Then $B_r(x)$ is contractible in $B_{2r}(x)$ for all $0<r\le\delta/10$.

2.2 Equivariant GH convergence with Ricci and volume lower bounds

In the study of fundamental groups associated to a convergent sequence

$$(M_i,x_i)\xrightarrow{GH}(X,x)$$

with conditions (2.1), it is natural to take the universal covers and their convergence into account. A powerful tool is the equivariant Gromov-Hausdorff convergence introduced by Fukaya-Yamaguchi [9]. After passing to a subsequence, we can obtain convergence

$$\begin{array}{ccc} (\widetilde{M}_i,\tilde{x}_i,\Gamma_i) & \xrightarrow{GH} & (Y,y,\Gamma) \\ \Big\downarrow \pi_i & & \Big\downarrow \pi \\ (M_i,x_i) & \xrightarrow{GH} & (X,x). \end{array} \tag{2.2}$$

Here $\Gamma_i = \pi_1(M_i,x_i)$ acts isometrically, freely, and discretely on the universal cover $(\widetilde{M}_i,\tilde{x}_i)$. This sequence of Γ_i-actions converges to a limit isometric Γ-action on the limit space Y. Due to the noncollapsing condition on (M_i,x_i), the limit group Γ is a discrete subgroup of $\mathrm{Isom}(Y)$; see Corollary 5.1.

We below state a result by Chen-Rong-Xu [6], which roughly states that if a point $z \in Y$ is sufficiently regular, then Γ-action cannot fix z.

Theorem 2.6 ([6, Theorem 2.1 and Corollary 2.2]). *Given $n,v>0$, there is a constant $\epsilon(n,v)>0$ such that the following holds.*

In the convergence (2.2) *with conditions* (2.1), *if $z \in Y$ is (ϵ,δ)-regular, where $\delta>0$, then Γ acts freely on $B_{\delta/4}(z)$.*

We will also need a quantitative result describing the action of any non-trivial subgroup of $\mathrm{Isom}(Y)$, which is proved in a joint work by Rong and the author [12]. Given a subgroup $H \le \mathrm{Isom}(Y)$, we write its displacement on a 1-ball by

$$D_{1,y}(H) = \sup\{d(hz,z) \mid z \in B_1(y), h \in H\}.$$

Theorem 2.7 ([12, Theorem 0.8]). *Given $n,v>0$, there is a constant $\delta(n,v)>0$ such that for any space $(Y,y) \in \mathcal{M}(n,-1,v)$ and any nontrivial subgroup of H of* $\mathrm{Isom}(X)$, *$D_{1,y}(H) \ge \delta(n,v)$ holds.*

3 Illustrative examples

In this short section, we briefly review some relevant examples of convergent sequences $M_i \overset{GH}{\to} X$ with conditions (1.1) by Otsu [11] and Anderson [1]. In particular, we shall see that in general the homomorphisms ϕ_i in Theorem 1.1 and ψ_i in Theorem 1.2 are not injective.

Example 3.1. Otsu [11] constructed a sequence of doubly warped metric products on $M = S^{p+1} \times S^q$, where $p \ge 2$ and $q \ge 2$:

$$[0,b_i] \times_{f_i} S^p \times_{h_i} S^q, \qquad g_i = dr^2 + f_i^2(r) ds_p^2 + h_i^2(r) ds_q^2.$$

such that

$$\mathrm{Ric}(g_i) \geq n-1, \qquad \mathrm{diam}(g_i) = b_i \to \pi, \qquad \mathrm{vol}(g_i) \geq v > 0.$$

At $s=0$ or b_i, f_i and g_i satisfies

$$f_i(s)=0, \qquad f_i'(s)=1, \qquad h_i(s)>0, \qquad \lim_{i\to\infty} h_i(s) \to 0 \qquad h_i'(s)=0.$$

As $i\to\infty$, (M,g_i) converges to $\mathrm{Susp}(S^p\times S^q)$, a suspension over $S^p\times S^q$.

Since the S^q-factor is always the round sphere in the construction, we can take the antipodal $\mathbb{Z}_2$-action on the S^q-factor and consider the quotient $(N_i,\bar{g}_i)=(M,g_i)/\mathbb{Z}_2$. The resulting $(N_i,\bar{g}_i)$ is Riemannian because the $\mathbb{Z}_2$-action is isometric and free on (M,g_i). Then as $i\to\infty$, N_i converges to $X=\mathrm{Susp}(S^p\times\mathbb{R}P^q)$. In terms of fundamental groups, we have

$$\pi_1(N_i)=\mathbb{Z}_2, \qquad \pi_1(X)=\mathrm{id}.$$

The forward homomorphism $\phi_i:\pi_1(N_i)\to\pi_1(X)$ has kernel $\mathbb{Z}_2$. The limit space X has two singular points as the vertices of the suspension. For small ϵ and $\delta>0$, $\mathcal{R}^\circ_{\epsilon,\delta}$ is homeomorphic to $(0,1)\times S^p\times\mathbb{R}P^q$. In particular, $\pi_1(\mathcal{R}^\circ_{\epsilon,\delta})=\mathbb{Z}_2$.

Example 3.2. Modifying the Eguchi-Hanson metric [8] on TS^2, the tangent bundle of S^2, Anderson [1] constructed a sequence of metrics g_i on M^4, the double of the disk bundle in TS^2, with

$$\mathrm{Ric}(g_i)\geq 0, \qquad \mathrm{diam}(g_i)\leq D, \qquad \mathrm{vol}(g_i)\geq v>0.$$

M is diffeomorphic to $S^2\times S^2$. Recall that the Eguchi-Hanson metric, written as h, on TS^2 is Ricci-flat and has Euclidean volume growth. It has a unique asymptotic cone as $C(\mathbb{R}P^3)=\mathbb{R}^4/\mathbb{Z}_2$.

Let $\mathcal{Z}$ be the zero-section in TS^2 and let $B_i=T_1(\mathcal{Z},r_i^{-2}h)$ be the tubular neighborhood of $\mathcal{Z}$ of radius 1 with respect to the metric $r_i^{-2}h$, where $r_i\to\infty$. Modifying the metric around ∂B_i and then doubling it, one obtains the desired metric g_i on M. As $i\to\infty$, (M,g_i) converges to $X=\mathrm{Susp}(\mathbb{R}P^3)$, a suspension over $\mathbb{R}P^3$. X has two singular points as the vertices. For small $\epsilon,\delta>0$, $\mathcal{R}^\circ_{\epsilon,\delta}$ is homeomorphic to $(0,1)\times\mathbb{R}P^3$. Hence

$$\pi_1(M)=\pi_1(X)=\mathrm{id}, \qquad \pi_1(\mathcal{R}^\circ_{\epsilon,\delta})=\mathbb{Z}_2.$$

The backward homomorphism $\psi_i:\pi_1(\mathcal{R}^\circ_{\epsilon,\delta})\to\pi_1(M)$ has kernel $\mathbb{Z}_2$.

4 Construction of ψ_i

In this section, we always assume that M_i is a sequence of closed n-manifolds with (1.1) that Gromov-Hausdorff converges to X. Let $0<\epsilon<\epsilon_0(n)/2$, where $\epsilon_0(n)$ is the constant in Theorem 2.5. Let x be a regular point of X and x_i in M_i converging to x. By the proof of Lemma 2.1(1), there is $\delta>0$ such that $B_\delta(x)\subseteq\mathcal{R}_{\epsilon,\delta}$, thus $x\in\mathcal{R}^\circ_{\epsilon,\delta}$. We may further shrink this δ later. The main goal of this section is to construct the group homomorphisms

$$\psi_i^\delta:\pi_1(\mathcal{R}^\circ_{\epsilon,\delta},x)\to\pi_1(M_i,x_i)$$

for all i large.

Lemma 4.1. *Given any $0<\epsilon<\epsilon_0(n)/2$ and $\delta>0$, the following holds for all large i.*

Let z_i be a point in M_i that is $\delta/30$-close to a point $z\in\mathcal{R}^\circ_{\epsilon,\delta}$. Then any loop in $B_{\delta/30}(z_i)$ is contractible in $B_\delta(z_i)$.

Proof. We set

$$\eta_i=d_{GH}(M_i,X)\to 0.$$

Then for each $z\in X$, we can choose a point $w_i\in M_i$ that is η_i-close to z. By the convergence $M_i\xrightarrow{GH}X$ and the compactness of X, there is i_0 large such that

$$d_{GH}(B_\delta(w_i),B_\delta(z))\le\frac{\epsilon_0(n)}{2}\delta$$

holds for all $z\in X$, all $w_i\in M_i$ that is η_i-close to z, and all $i\ge i_0$, where $\epsilon_0(n)$ is the constant in Theorem 2.5. Now fixing a point $z\in\mathcal{R}^\circ_{\epsilon,\delta}$, we have

$$d_{GH}(B_\delta(z),B^n_\delta(0))\le\epsilon\delta.$$

Thus by triangle inequality,

$$d_{GH}(B_\delta(w_i),B^n_\delta(0))\le(\epsilon+\epsilon_0(n)/2)\delta<\epsilon_0(n)\delta.$$

Then by Theorem 2.5, every loop in $B_{\delta/10}(w_i)$ is contractible in $B_{\delta/5}(w_i)$. Let z_i be any point in M_i that is $\delta/30$-close to z. We have

$$d(z_i,w_i)\le d(z_i,z)+d(z,w_i)\le\delta/30+\eta_i.$$

Thus when i is large with $\eta_i<\delta/30$, we see that $B_{\delta/30}(z_i)\subseteq B_{\delta/10}(w_i)$. Therefore, every loop in $B_{\delta/30}(z_i)$ is contractible in $B_{\delta/5}(w_i)\subseteq B_\delta(z_i)$. □

With Lemma 4.1, we follow a similar construction in [13, Lemma 2.4] (also see [15]) to construct nearby loops and homotopies on M_i from the ones on $\mathcal{R}^\circ_{\epsilon,\delta}$. For two compact length metric spaces (X_1,x_1) and (X_2,x_2) that are close in the Gromov-Hausdorff distance, we say that two curves $\sigma_j:[0,1]\to X_j$, where $j=1,2$, are ϵ-close, if

$$d(\sigma_1(t),\sigma_2(t))\le\epsilon$$

for all $t\in[0,1]$; in other words, $\sigma_1(t)\in X_1$ is ϵ Gromov-Hausdorff close to $\sigma_2(t)\in X_2$ for all $t\in[0,1]$.

Lemma 4.2. *We write $\eta_i = d_{GH}(M_i,X) \to 0$. Then for sufficiently large i, the followings hold.*

(1) *For any loop $\sigma:[0,1]\to\mathcal{R}^\circ_{\epsilon,\delta}$, there is a loop σ_i in M_i that is $5\eta_i$-close to σ.*

(2) *Let σ_i and σ_i' be loops in M_i that are both $\delta/300$-close to a loop σ in $\mathcal{R}^\circ_{\epsilon,\delta}$, then σ_i and σ_i' are free homotopic in M_i.*

(3) *Let σ and τ be two loops in $\mathcal{R}^\circ_{\epsilon,\delta}$. Let σ_i and τ_i be loops in M_i that is $\delta/300$-close to σ and τ, respectively. If σ and τ are free homotopic in $\mathcal{R}^\circ_{\epsilon,\delta}$, then σ_i and τ_i are free homotopic in M_i.*

Proof. (1) The construction of σ_i is the same as the proof of [13, Lemma 2.4 (1)]. Namely, using the uniform continuity of σ, we choose a suitable partition of $[0,1]$. Then for each intermediate point in the partition, we can pick nearby points in M_i and then join them by minimal geodesics. (2) By uniform continuity of σ, we choose $l>0$ such that

$$d(\sigma(t),\sigma(t'))\leq\delta/300$$

for all $t,t'\in[0,1]$ with $|t-t'|\leq l$. Let $\{t_0=0,t_1,...,t_j,...,t_N=1\}$ be a partition of $[0,1]$ with $|t_{j+1}+t_j|\leq l$ for all j. By triangle inequality, it is clear that

$$d(\sigma_i(t_j),\sigma_i(t_{j+1}))\leq 3\cdot\delta/300,\qquad d(\sigma_i'(t_j),\sigma_i'(t_{j+1}))\leq 3\cdot\delta/300.$$

Let $c_{i,j}$ be the loop obtained by joining $\sigma_i|_{[t_j,t_{j+1}]}$, a minimal geodesic from $\sigma_i(t_{j+1})$ to $\sigma_i'(t_{j+1})$, the inverse of $\sigma_i'|_{[t_j,t_{j+1}]}$, and lastly a minimal geodesic from $\sigma_i'(t_j)$ to $\sigma_i(t_j)$. Since

$$d(\sigma_i(t_j),\sigma_i'(t_j))\leq 2\cdot\delta/300$$

for all i. By construction, one can verify that

$$\text{image of } c_{i,j}\subseteq B_{\delta/30}(\sigma_i(t_j)).$$

Because $\sigma_i(t_j)$ is $\delta/300$-close to $\sigma(t_j)\in\mathcal{R}^\circ_{\epsilon,\delta}$, by Lemma 4.1, $c_{i,j}$ is contractible in M_i for all j. Thus σ_i and σ_i' are free homotopic. (3) Let $H:S^1\times[0,1]\to\mathcal{R}^\circ_{\epsilon,\delta}$ be a homotopy between σ and τ. We follow the method in [13, Lemma 2.4] to construct a homotopy H_i between σ_i and τ_i as below. By the uniform continuity of H, we can choose a finite triangular decomposition Σ of $S^1\times[0,1]$ so that

$$\operatorname{diam}(H(\Delta))\leq\delta/300$$

for each triangle Δ of Σ. For any vertex v of Σ, if v is on the boundary of $S^1\times[0,1]$, then $H_i(v)$ is naturally defined as a point on σ_i or τ_i; if not, then we define $H_i(v)$ as a point in M_i that is η_i-close to $H(v)$. Next, we define H_i on every edge of Σ: for an edge that is on the boundary of $S^1\times[0,1]$, H_i on this edge is naturally defined as part of σ_i or τ_i;

for an edge not on the boundary with vertices v and w, we map it to a minimal geodesic between $H_i(v)$ and $H_i(w)$. If $\eta_i \le \delta/300$, then by construction, every triangle Δ satisfies

$$H_i(\partial\Delta) \subseteq B_{\delta/30}(H_i(v)),$$

where v is a vertex of Δ. Since $H_i(v)$ is $\delta/300$-close to $H(v) \in \mathcal{R}^\circ_{\epsilon,\delta}$, we can apply Lemma 4.1 to contract the loop $H_i(\Delta)$. Applying this to all the triangles of Σ, we result in the desired homotopy between σ_i and τ_i. □

Now we construct the backward homomorphism ψ_i^δ.

Definition 4.1. *Let $[\sigma] \in \pi_1(\mathcal{R}^\circ_{\epsilon,\delta}, x)$ represented by a loop σ based at x in $\mathcal{R}^\circ_{\epsilon,\delta}$. For i large that fulfills Lemma 4.2, we draw a loop σ_i in M_i based at x_i that is $\delta/300$-close to σ. We define*

$$\begin{aligned} \psi_i^\delta : \pi_1(\mathcal{R}^\circ_{\epsilon,\delta}, x) &\to \pi_1(M_i, x_i), \\ [\sigma] &\mapsto [\sigma_i]. \end{aligned}$$

Theorem 4.1. *The above constructed ψ_i^δ is well-defined and is a group homomorphism for all i large.*

Proof. By Lemma 4.2 (2), $\psi_i^\delta[\sigma] = [\sigma_i]$ is independent of the choice of σ_i. It also follows from Lemma 4.2 (3) that the definition is independent of the choice of σ.

It is straightforward to check that ψ_i^δ is a group homomorphism. In fact, let σ and τ be two loops in $\mathcal{R}^\circ_{\epsilon,\delta}$ based at x, and let σ_i and τ_i be loops in M_i that is $\delta/300$-close to σ and τ, respectively. Since the the product $\sigma_i \cdot \tau_i$ is clearly $\delta/300$-close to $\sigma \cdot \tau$, by definition, we have

$$\psi_i^\delta[\sigma] \cdot \psi_i^\delta[\tau] = [\sigma_i] \cdot [\tau_i] = [\sigma_i \cdot \tau_i] = \psi_i^\delta[\sigma \cdot \tau] = \psi_i^\delta([\sigma] \cdot [\tau]).$$ □

For $0 < \epsilon \le \epsilon'$ and $0 < \delta' \le \delta$, we have inclusion

$$\mathcal{R}^\circ_{\epsilon,\delta} \subseteq \mathcal{R}^\circ_{\epsilon',\delta'}.$$

For both $\mathcal{R}^\circ_{\epsilon,\delta}$ and $\mathcal{R}^\circ_{\epsilon',\delta'}$, we have backward homomorphisms defined; they are indeed related by the inclusion map, as stated in Lemma 5.3 below. Due to the dependence on ϵ, we will write $\psi_i^{\epsilon,\delta}$ instead of ψ_i^δ for clarity.

Lemma 4.3. *Let $0 < \epsilon \le \epsilon' < \epsilon_0(n)/2$ and $0 < \delta' \le \delta$. Suppose that i is large such that both homomorphisms*

$$\psi_i^{\epsilon,\delta} : \pi_1(\mathcal{R}^\circ_{\epsilon,\delta}, x) \to \pi_1(M_i, x_i), \quad \psi_i^{\epsilon',\delta'} : \pi_1(\mathcal{R}^\circ_{\epsilon',\delta'}, x) \to \pi_1(M_i, x_i)$$

are defined. Then $\psi_i^{\epsilon,\delta}$ coincides with the composition

$$\pi_1(\mathcal{R}^\circ_{\epsilon,\delta}, x) \xrightarrow{\iota_\star} \pi_1(\mathcal{R}^\circ_{\epsilon',\delta'}, x) \xrightarrow{\psi_i^{\epsilon',\delta'}} \pi_1(M_i, x_i),$$

where ι is the inclusion map $\mathcal{R}^\circ_{\epsilon,\delta} \hookrightarrow \mathcal{R}^\circ_{\epsilon',\delta'}$.

Proof. Let $[\sigma]\in\pi_1(\mathcal{R}^\circ_{\epsilon,\delta},x)$, where σ is a loop in $\mathcal{R}^\circ_{\epsilon,\delta}$ based at x. Then $\iota\circ\sigma$ naturally represents an element of $\pi_1(\mathcal{R}^\circ_{\epsilon',\delta'},x)$. Let σ_i be a loop in M_i based at x_i that is $\delta'/300$-close to $\iota\circ\sigma$. According to Definition 4.1, we have

$$\psi_i^{\epsilon',\delta'}\circ\iota_\star[\sigma]=\psi_i^{\epsilon',\delta'}[\iota\circ\sigma]=[\sigma_i].$$

Since $\delta'\le\delta$, the loop σ_i is also $\delta/300$-close to $\iota\circ\sigma=\sigma$ in $\mathcal{R}^\circ_{\epsilon,\delta}$. Therefore,

$$\psi_i^{\epsilon,\delta}[\sigma]=[\sigma_i]=\psi_i^{\epsilon',\delta'}\circ\iota_\star[\sigma].$$

□

5 Surjectivity of ψ_i

The main goal of this section is to prove Theorem 1.2. The proof of surjectivity of ψ_i^δ is a contradicting argument and we shall apply equivariant GH convergence to the contradicting sequence.

Before starting the proof of Theorem 1.2, we prove some results about the equivariant GH convergence.

Lemma 5.1. *Let us consider the diagram* (2.2) *with conditions* (2.1)*. Suppose that $x\in\mathcal{R}^\circ_{\epsilon,\delta}$, where $0<\epsilon\le\epsilon(n)$ and $0<\delta\le\delta(n)$ are sufficiently small. Then there is a constant $l(n,\delta)>0$ such that any nontrivial element in $\pi_1(M_i,x_i)$ has length at least $l(n,\delta)$, where i is large.*

Proof. The proof is a localized version of an argument by Anderson [1].

Let $g_i\in\pi_1(M_i,x_i)$ with $d(g_i\tilde{x}_i,\tilde{x}_i)=l_i>0$. We shall prove a lower bound for $\liminf l_i:=l$. Let F_i be the Dirichlet domain of $\widetilde{M}_i$ centered at $\tilde{x}_i$. Since

$$g_i(F_i\cap B_\delta(\tilde{x}_i))\subseteq B_{l_i+\delta}(\tilde{x}_i),\qquad g_i(F_i\cap B_\delta(\tilde{x}_i))\cap(F_i\cap B_\delta(\tilde{x}_i))=\emptyset,$$

we have volume estimate

$$\begin{aligned}2\mathrm{vol}(B_\delta(x_i))&=\mathrm{vol}(F_i\cap B_\delta(\tilde{x}_i))+\mathrm{vol}(g_i(F_i\cap B_\delta(\tilde{x}_i)))\\&\le\mathrm{vol}(B_{l_i+\delta}(\tilde{x}_i))\\&\le v(n,-1,l_i+\delta),\end{aligned}$$

where $v(n,-\kappa,r)$ means the volume of an r-ball in the n-dimensional space form of constant curvature $-\kappa$. By volume convergence, as $i\to\infty$, we have

$$\begin{aligned}\mathrm{vol}(B_\delta(x_i))&\to\mathcal{H}^n(B_\delta(x))\\&\ge(1-\Psi(\epsilon|n))\cdot v(n,0,\delta)\\&\ge(1-\Psi(\epsilon|n))\cdot(1-\Psi(\delta|n))\cdot v(n,-1,\delta).\end{aligned}$$

These lead to

$$\frac{v(n,-1,l_i+\delta)}{v(n,-1,\delta)}\ge1.9(1-\Psi(\epsilon,\delta|n))>1.5.$$

for all i large, which gives a universal lower bound $l(n,\delta)$ for $\liminf l_i$. □

Corollary 5.1. *In the diagram* (2.2) *with conditions* (2.1)*, the limit group* Γ *is discrete.*

Proof. Let $z\in X$ be a regular point. We choose small $0<\epsilon<\epsilon(n)$ and $\delta>0$ such that $z\in\mathcal{R}^{\circ}_{\epsilon,\delta}$. Let $z_i\in M_i$ converging to z and let $\tilde{z}_i\in\widetilde{M}_i$ be a lift of z_i. By Lemma 5.1, the orbit $\Gamma_i\cdot\tilde{z}_i$ is $l(n,\delta)$-discrete. Passing this to the limit, we see that Γ is a discrete group. □

Lemma 5.2. *Let* $(N_i,x_i)\in\mathcal{M}(n,-1,v)$ *with an isometric* Γ_i*-action on each* N_i*. Suppose that the sequence converges*

$$(N_i,x_i,\Gamma_i)\xrightarrow{GH}(Y,y,G)$$

and the limit group G *is discrete. Let* $g\in G$ *be an element of finite order* k *and let* $\gamma_i\in\Gamma_i$ *converging to* g*. Then*
(1) γ_i *has order* k *for all* i *large;*
(2) $\langle\gamma_i\rangle\xrightarrow{GH}\langle g\rangle$*, where* $\langle\cdot\rangle$ *means the subgroup generated by that element.*

Proof. (1) First note that $\gamma_i^k\xrightarrow{GH}g^k=e$ as $i\to\infty$. We claim that $\langle\gamma_i^k\rangle\xrightarrow{GH}\{e\}$. In fact, let H be the limit of $\langle\gamma_i^k\rangle$ and suppose that H has a non-identity element h. We pick a point $z\in Y$ with $d(hz,z)>0$. Since $d(\gamma_i^k z_i,z_i)\to 0$, where $z_i\in M_i$ converging to z, for any $0<l<d(hz,z)$, we can find a sequence m_i such that

$$d((\gamma_i^k)^{m_i}z_i,z_i)\to l.$$

The sequence $(\gamma_i^k)^{m_i}$ would converge to an element of H with displacement l at z. Because $l\in(0,d(hz,z))$ is arbitrary, we result in a contradiction to the discreteness of G. This proves the claim.

By this claim, we have $D_{1,x_i}(\langle\gamma_i^k\rangle)\to 0$. On the other hand, by Theorem 2.7

$$D_{1,x_i}(\langle\gamma_i^k\rangle)\geq\delta(n,v)>0$$

if $\langle\gamma_i^k\rangle$ is nontrivial. We conclude that $\gamma_i^k=e$. It is clear that γ_i cannot have order m strictly less than k; otherwise $\gamma_i^m\xrightarrow{GH}e\neq g^m$. We complete the proof that γ_i has order k. (2) is a direct consequence of (1). □

Let γ be an isometry of Y. We write

$$\mathrm{Fix}(\gamma)=\{z\in Y|\gamma z=z\}$$

as the fixed point set of γ.

Proposition 5.1. *In the convergence* (2.2) *with conditions* (1.1), $\mathrm{Fix}(\gamma)$ *has Hausdorff dimension at most* $n-2$ *for all non-identity* $\gamma\in\Gamma$.

Proof. Suppose the contrary $\dim_{\mathcal{H}}(\mathrm{Fix}(\gamma)) > n-2$. Then $\mathcal{H}^l(\mathrm{Fix}(\gamma)) > 0$ for some real number $n-2 < l < n$. Let $\mathcal{S}$ be the singular set of Y. By Theorem 2.3, $\mathcal{C} := \mathrm{Fix}(\gamma) - \mathcal{S}$ also satisfies $\mathcal{H}^l(\mathcal{C}) > 0$. Let z be an l-density point of $\mathcal{C}$, that is, $z \in \mathcal{R} \cap \mathrm{Fix}(\gamma)$ such that

$$\limsup_{r\to 0} \frac{\mathcal{H}^l_\infty(\mathcal{C}\cap B_r(z))}{\omega_l r^l} \geq 2^{-l}.$$

Let $r_j \to \infty$ be a sequence that realizes the above limsup and let

$$(r_j Y, z) \xrightarrow{GH} (C_z Y = \mathbb{R}^n, v)$$

be a corresponding tangent cone at z. With respect to this convergent sequence, γ subconverges to a limit isometry g of $\mathbb{R}^n$, and $\mathcal{C}$ subconverges to a closed subset $\mathcal{C}_z \subseteq \mathbb{R}^n$. It is clear that by construction, g fixes every point in $\mathcal{C}_z$. By a standard covering argument, $\mathcal{C}_z \cap B_1(v)$, and thus $\mathrm{Fix}(g) \cap B_1(v)$, have positive l-dimensional Hausdorff measure.

The limit isometry g, which an isometry of $\mathbb{R}^n$, satisfies $\dim_{\mathcal{H}}(\mathrm{Fix}(g)) > n-2$. Since $g \in O(n)$, by linear algebra we conclude that g must be a reflection of $\mathbb{R}^n$ that fixes a hyperplane. In particular, g has order 2. By Lemma 5.2 (1), γ has order 2 as well. Let $\gamma_i \in \Gamma_i$ that converges to γ and let $\overline{M_i} = \widetilde{M_i}/\langle\gamma_i\rangle$. Lemma 5.2 allows us to consider the convergence

$$\begin{array}{ccccccc}
(\widetilde{M}_i, z_i, \langle\gamma_i\rangle) & \xrightarrow{GH} & (Y, z, \langle\gamma\rangle) & \qquad & (r_j Y, z, \langle\gamma\rangle) & \xrightarrow{GH} & (\mathbb{R}^n, v, \langle g\rangle) \\
\downarrow \pi_i & & \downarrow \pi & & \downarrow \pi_i & & \downarrow \pi \\
(\overline{M}_i, \bar{z}_i) & \xrightarrow{GH} & (\overline{Y} = Y/\langle\gamma\rangle, \bar{z}), & & (r_j \overline{Y}, \bar{z}) & \xrightarrow{GH} & (\mathbb{R}^n/\langle g\rangle, \bar{v}).
\end{array} \tag{5.1}$$

Because g is a reflection in $\mathbb{R}^n$, the quotient $\mathbb{R}^n/\langle g\rangle$ is isometric to the Euclidean halfspace $\mathbb{H}^n = \{(a_1,\ldots,a_n) | a_n \geq 0\}$. In particular, $\mathbb{H}^n$ appears as a tangent cone of a non-collapsing Ricci limit space $\overline{Y}$ at $\bar{y}$. This is a contradiction to Theorem 2.3 and thus completes the proof. □

If one seeks a weaker statement of Theorem 1.2 that requires $\epsilon(n,v)$ instead of $\epsilon(n)$, there is an alternative and shorter proof of Theorem 5.1 based on a result by Chen-Rong-Xu [6], that is, Theorem 2.6 which we have recalled in Section 2. We also include this short proof here since it may have some independent interest.

Proof. We shall show that $\mathrm{Fix}(\gamma) \subseteq \mathcal{S}$; then the Hausdorff dimension estimate follows from Theorem 2.3. In fact, let $z \in Y$ be a regular point and let $0 < \epsilon < \epsilon(n,v)$, the constant in Theorem 2.6. Then there is some $\delta > 0$ such that z is (ϵ,δ)-regular. Applying Theorem 2.6 to the first diagram of (5.1), we conclude that $\langle\gamma\rangle$-action, and thus γ, does not fix z. □

We are in a position to prove Theorem 1.2. For reader's convenience, we restate the surjectivity part in Theorem 1.2 as below.

Theorem 5.1. *Let $\psi_i^\delta : \pi_1(\mathcal{R}^\circ_{\epsilon,\delta}, x) \to \pi_1(M_i, x_i)$ be the group homomorphism constructed in Definition* 4.1*. When δ is sufficiently small, ψ_i^δ is surjective for all i large.*

Proof. We argue by contradiction. Suppose that for each $1/j$, where $j \in \mathbb{N}$, we can find some $i(j) \geq i$ and some element $g_{i(j)} \in \pi_1(M_{i(j)}, x_{i(j)})$ such that $g_{i(j)}$ is not in the image of $\psi_{i(j)}^{1/j}$. Since $\operatorname{diam}(M_i) \leq D$, $\pi_1(M_i, x_i)$ can be generated by elements of length at most $2D$. Together with Lemma 5.1, without loss of generality, we will assume that each $g_{i(j)}$ has length between $l(n,\delta)$ and $2D$ at $x_{i(j)}$.

For this sequence $i(j)$, after passing to a subsequence if necessary, we consider the equivariant Gromov-Hausdorff convergence:

$$\begin{array}{ccc} (\widetilde{M}_{i(j)}, \tilde{x}_{i(j)}, \Gamma_{i(j)}, g_{i(j)}) & \xrightarrow{GH} & (Y, y, \Gamma, g) \\ \Big\downarrow \pi_i & & \Big\downarrow \pi \\ (M_{i(j)}, x_{i(j)}) & \xrightarrow{GH} & (X, x). \end{array}$$

Because x is regular, so is y. Under the isometry g, gy is regular as well with $d(gy,y) \in [l(n,\delta), 2D]$. By Lemma 5.1, the points y and gy are not fixed by any $\gamma \in \Gamma - \{e\}$ because they are lifts of $x \in \mathcal{R}^\circ_{\epsilon,\delta}$. Let

$$\mathcal{C} = (Y - \mathcal{R}^\circ_\epsilon(Y)) \cup \left(\bigcup_{\gamma \in \Gamma - \{e\}} \operatorname{Fix}(\gamma) \right).$$

By Theorem 2.3, $\mathcal{C}$ has Hausdorff dimension at most $n-2$.

We claim that $\mathcal{C}$ is closed. It suffices to show that $\cup_{\gamma \in \Gamma - \{e\}} \operatorname{Fix}(\gamma)$ is closed. In fact, let z_i be a convergent sequence $\cup_{\gamma \in \Gamma - \{e\}} \operatorname{Fix}(\gamma)$ with limit z. Each z_i is fixed by some element $\gamma_i \in \Gamma - \{e\}$. Because each γ_i, where i large, moves z at most by distance 1, γ_i is precompact in Γ. By the discreteness of Γ, we see that all γ_i are the same after passing to a subsequence: $\gamma_i = g \in \Gamma - \{e\}$. Hence

$$gz = \lim_{i \to \infty} g z_i = \lim_{i \to \infty} z_i = z.$$

This shows that $z \in \cup_{\gamma \in \Gamma - \{e\}} \operatorname{Fix}(\gamma)$. As a result, $\cup_{\gamma \in \Gamma - \{e\}} \operatorname{Fix}(\gamma)$ is closed.

We note that $y, gy \in Y - \mathcal{C}$ because they are in $\pi^{-1}(x)$ and thus not fixed by any $\gamma \in \Gamma - \{e\}$ according to Lemma 5.1. As a result of Theorem 2.4, we can connect y and gy by a path σ that is contained in $Y - \mathcal{C}$. In particular, σ is in the regular set and avoids any point that is fixed by some nontrivial element of Γ. Because the image of σ is compact and $\mathcal{C}$ is closed. The distance $\delta_1 = d(\sigma, \mathcal{C})$ is positive. Let

$$T := T_{\delta_1/2}(\sigma) = \{ z \in Y \mid d(z,\sigma) \leq \delta_1/2 \}$$

be the closed tubular neighborhood of σ with radius $\delta_1/2$. By construction, T does not intersect $\mathrm{Fix}(\gamma)$ for all non-identity $\gamma\in\Gamma$. Because T is compact,

$$\delta_2 := \inf_{a\in T,\gamma\in\Gamma-\{e\}} d(a,\gamma a)$$

is positive. Setting $\delta_3=\min\{\delta_1/2,\delta_2/4\}$, we claim that $B_{\delta_3}(z)$ is isometric to $B_{\delta_3}(\pi(z))\subseteq X$ for all $z\in\sigma$, where $\pi: Y\to X=Y/\Gamma$ is the quotient map. In fact, first note that for any two points $a,b\in B_{\delta_3}(z)$, we clearly have $a,b\in T$. Then for any other orbit point $a'\in\Gamma a-\{a\}$, it follows from triangle inequality that

$$d(a',b)\geq d(a',a)-d(a,b)\geq \delta_2-2\delta_3\geq \delta_2/2>d(a,b).$$

This verifies the claim: for all $a,b\in B_{\delta_3}(z)$,

$$d_Y(a,b)=d_Y(\Gamma a,\Gamma b)=d_X(\pi(a),\pi(b)).$$

We choose a small $\epsilon_1>0$ such that $\Psi(\epsilon_1|n)\leq\epsilon$, where Ψ is the function in Lemma 2.1(1). With this ϵ_1, by Lemma 2.1(2), there is $\delta_4>0$ such that

$$\sigma\subseteq\mathcal{R}_{\epsilon_1,\delta_4}(Y).$$

Let $\delta_5:=\min\{\delta_3,\delta_4\}>0$. Since $B_{\delta_5}(z)$ is isometric to $B_{\delta_5}(\pi(z))$ for all $z\in\sigma$, together with Lemma 2.1 (1), we conclude that

$$\overline{\sigma}:=\pi(\sigma)\subseteq\mathcal{R}_{\epsilon_1,\delta_5}(X)\subseteq\mathrm{Int}\mathcal{R}_{\epsilon,\delta_5/3}.$$

Now we go back to the sequence of manifolds. Along $\widetilde{M}_{i(j)}$, let $\sigma_{i(j)}$ be a sequence of paths from $\tilde{x}_{i(j)}$ to $g_{i(j)}\tilde{x}_{i(j)}$ that converges uniformly to σ. Then its projection $\pi_{i(j)}(\sigma_{i(j)})=:\overline{\sigma_{i(j)}}$ is a loop that represents $g_{i(j)}$ and uniformly converges to a loop $\pi(\sigma)$ in X as $j\to\infty$. By the construction in Definition 4.1, when j is large we have

$$\psi_{i(j)}^{\delta_5/3}:\pi_1(\mathrm{Int}\mathcal{R}_{\epsilon,\delta_5/3},x)\to\pi_1(M_i,x_i)\ \text{ with }\ \psi_{i(j)}^{\delta_5/3}[\overline{\sigma_{i(j)}}]=g_{i(j)}.$$

Applying Lemma 4.3 with $\epsilon=\epsilon'$, we obtain

$$g_{i(j)}=\psi_{i(j)}^{\delta_5/3}[\overline{\sigma}]=\psi_{i(j)}^{1/j}\circ\iota_\star[\overline{\sigma}],$$

where ι is the inclusion map $\mathrm{Int}\mathcal{R}_{\epsilon,\delta_5/3}\hookrightarrow\mathrm{Int}\mathcal{R}_{\epsilon,1/j}$. In particular, $g_{i(j)}$ is in the image of $\psi_{i(j)}^{1/j}$. This contradicts with our choice in the beginning that $g_{i(j)}$ is not in the image of $\psi_{i(j)}^{1/j}$ and thus completes the proof. $\square$

With Theorems 4.1 and 5.1, now we complete the proof of Theorem 1.2 by Lemma 5.3 below.

Lemma 5.3. *Let*

$$\phi_i : \pi_1(M_i, x_i) \to \pi_1(X, x), \quad \psi_i^\delta : \pi_1(\mathcal{R}_{\epsilon,\delta}^\circ, x) \to \pi_1(M_i, x_i)$$

be the surjective homomorphisms in Theorems 1.1 *and* 5.1, *respectively. Then*

$$\phi_i \circ \psi_i^\delta : \pi_1(\mathcal{R}_{\epsilon,\delta}^\circ, x) \to \pi_1(X, x)$$

coincides with $\iota_\star$ for all i large, where $\iota : \mathcal{R}_{\epsilon,\delta}^\circ \hookrightarrow X$ is the inclusion map.

Proof. Because X is semi-locally simply connected [13], there is $\delta_0 > 0$ such that every loop contained in a δ_0-ball of X is contractible in X. We set

$$\delta_1 = \min\{\delta_0/20, \delta/300\}.$$

We recall that the forward homomorphism ϕ_i can be constructed as follows ([15] or [13] for details). When i is large such that $d_{GH}(M_i, X) \le \delta_1$, for any loop σ_i in M_i based at x_i, we can draw a loop σ in X based at x such that σ is $5\delta_1$-close to σ_i. Then one can define the desired ϕ_i by sending $[\sigma_i]$ to $[\sigma]$. The choice of δ_0 assures that ϕ_i is well-defined and a surjective homomorphism.

Now let $[\sigma] \in \pi_1(\mathcal{R}_{\epsilon,\delta}^\circ, x)$ represented by a loop σ based at x in $\mathcal{R}_{\epsilon,\delta}^\circ$. When i is large, let σ_i be a loop based at $x_i \in M_i$ that is δ_1-close to σ. By the constructions of ϕ_i and ψ_i^δ, we have

$$\phi_i \circ \psi_i^\delta [\sigma] = \phi_i[\sigma_i] = [\sigma] \in \pi_1(X, x). \qquad \square$$

Next, we prove Theorem 1.3.

Proof of Theorem 1.3. We choose a sufficiently small $\delta > 0$ so that we can apply Theorem 1.2 to construct surjective group homomorphisms

$$\psi_i^\delta : \pi_1(\mathcal{R}_{\epsilon,\delta}^\circ, x) \to \pi_1(M_i, x_i)$$

for all i large. If the inclusion map $\iota^\delta : \mathcal{R}_{\epsilon,\delta}^\circ \hookrightarrow X$ induces an injective homomorphism

$$\iota_\star^\delta : \pi_1(\mathcal{R}_{\epsilon,\delta}^\circ, x) \to \pi_1(X, x),$$

then by Theorem 1.2 (2), the composition

$$\pi_1(\mathcal{R}_{\epsilon,\delta}^\circ, x) \xrightarrow{\psi_i^\delta} \pi_1(M_i, x_i) \xrightarrow{\phi_i} \pi_1(X, x)$$

is an isomorphism. Together with the surjectivity of ψ_i^δ and ϕ_i, we clearly have isomorphism $\pi_1(X) \simeq \pi_1(M_i)$. In general, if $\iota_\star^\delta$ is not injective, we shall analyze its kernel. We claim that

$$\ker \iota_\star^\delta = \ker \psi_i^\delta.$$

If this claim holds, then

$$\pi_1(M_i,x_i)=\frac{\pi_1(\mathcal{R}^\circ_{\epsilon,\delta},x)}{\ker\psi_i^\delta}=\frac{\pi_1(\mathcal{R}^\circ_{\epsilon,\delta},x)}{\ker\iota_\star^\delta}=\pi_1(X,x).$$

One side of the inclusion $\ker\psi_i^\delta\subseteq\ker\iota_\star^\delta$ is clear due to Theorem 1.2 (2). It remains to prove the other direction. Let us consider a composition of inclusion maps $\iota\circ j=\iota^\delta$:

$$\mathcal{R}^\circ_{\epsilon,\delta}\overset{j}{\hookrightarrow}\mathcal{R}^\circ_\epsilon\overset{\iota}{\hookrightarrow}X.$$

They induce

$$\pi_1(\mathcal{R}^\circ_{\epsilon,\delta},x)\xrightarrow{j_\star}\pi_1(\mathcal{R}^\circ_\epsilon,x)\xrightarrow{\iota_\star}\pi_1(X,x)$$

with $\iota_\star\circ j_\star=\iota_\star^\delta$ being surjective. By the assumption that $\iota_\star$ is injective, $\iota_\star$ is an isomorphism and

$$\ker\iota_\star^\delta=\ker j_\star.$$

Let $[\sigma]\in\ker j_\star$ represented by a loop σ at x in $\mathcal{R}^\circ_{\epsilon,\delta}$. Then σ is contractible in $\mathcal{R}^\circ_\epsilon$. Let $H:[0,1]^2\to\mathcal{R}^\circ_\epsilon$ be a nullhomotopy of σ. By Lemma 2.1 (1,2), there are $\epsilon'=\Psi(\epsilon|n)>\epsilon$ and $0<\delta'<\delta$ such that

$$H([0,1]^2)\subseteq\mathcal{R}^\circ_{\epsilon',\delta'}.$$

When i is large, we draw a loop σ_i based at $x_i\in M_i$ that is $\delta'/300$-close to σ. It follows from Lemma 4.2 (3) that σ_i is contractible in M_i. By the construction of

$$\psi_i^{\epsilon',\delta'}:\pi_1(\mathcal{R}^\circ_{\epsilon',\delta'},x)\to\pi_1(M_i,x_i)$$

and Lemma 5.3, we have

$$\psi_i^{\epsilon,\delta}[\sigma]=\psi_i^{\epsilon',\delta'}[\sigma]=[\sigma_i]=\mathrm{id}\in\pi_1(M_i,x_i).$$

This shows that

$$\ker\psi_i^{\epsilon,\delta}\supseteq\ker j_\star=\ker\iota_\star^\delta$$

and hence completes the proof.

When $\mathcal{R}^\circ_\epsilon$ is simply connected, because $\iota_\star$ is an isomorphism as shown above, X is also simply connected. Consequently, $\pi_1(M_i)\simeq\pi_1(X)$ is simply connected. $\square$

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