J. Math. Study
doi: 10.4208/jms.v58n1.25.02

Vol. **58**, No. 1, pp. 22-37
March 2025

Quasi-Convex Subsets and the Farthest Direction in Alexandrov Spaces with Lower Curvature Bound

Xiaole Su[1], Hongwei Sun[2] and Yusheng Wang[1,*]

[1] *School of Mathematical Sciences & Lab. Math. Com. Sys., Beijing Normal University, Beijing 100875, China;*
[2] *School of Mathematical Sciences, Capital Normal University, Beijing 100037, China.*

Received April 17, 2024; Accepted December 4, 2024;
Published online March 30, 2025.

In honor of Professor Xiaochun Rong on his seventieth birthday

---

Abstract. Let F be a closed subset in a finite dimensional Alexandrov space X with lower curvature bound. This paper shows that F is quasi-convex if and only if, for any two distinct points $p,r\in F$, if there is a direction at p which is more than $\frac{\pi}{2}$ away from $\Uparrow_p^r$ (the set of all directions from p to r), then the farthest direction to $\Uparrow_p^r$ at p is tangent to F. This implies that F is quasi-convex if and only if the gradient curve starting from r of the distance function to p lies in F. As an application, we obtain that the fixed point set of an isometry on X is quasi-convex.

AMS subject classifications: 53C20, 51F99

Key words: Quasi-convex subset, Alexandrov space, extremal subset, gradient curve.

---

1 Introduction

Finite dimensional Alexandrov spaces with lower curvature bound can be viewed as a generalization of Riemannian manifolds with lower sectional curvature bound [2]. Compared to Riemannian manifolds, Alexandrov spaces might have some singularities, so that some important subsets appear as some kinds of analogues of totally geodesic submanifolds, such as convex subsets, extremal subsets and quasi-geodesics [3,4]. Recently, such a kind of subsets named quasi-convex subsets has been introduced [6]. They include not only all convex subsets without boundary and extremal subsets but also more other subsets, such as fixed point sets of isometries on Alexandrov spaces. Moreover, all shortest pathes in a quasi-convex subset are quasi-geodesics.

*Corresponding author. *Email addresses:* suxiaole@bnu.edu.cn (Su X), 5598@cnu.edu.cn (Sun H), wyusheng@bnu.edu.cn (Wang Y)

To show the definition of quasi-convex subsets, we first make some conventions on notations.

- $\mathrm{Alex}(k)$: the set of complete and finite dimensional Alexandrov spaces with curvature $\geqslant k$.
- $\mathbb{S}_k^2$: the complete and simply connected space form of dimension 2 and curvature k.
- $|pq|$, $[pq]$: the distance, a minimal geodesic (i.e. shortest path) between p and q.
- $\uparrow_p^q$: the direction from p to q for a given $[pq]$.
- $\Uparrow_p^q$: the set of all directions from p to q.

For a point $p\in X\in\mathrm{Alex}(k)$, we denote by Σ_pX the space of directions of X at p which belongs to $\mathrm{Alex}(1)$ [2] . In Σ_pX, $\Uparrow_p^q$ is a closed subset, and $|\Uparrow_p^q\Uparrow_p^r|$ is the distance between $\Uparrow_p^q$ and $\Uparrow_p^r$. And to $p,q,r\in X$, we associate $\tilde p,\tilde q,\tilde r\in\mathbb{S}_k^2$ with $|\tilde p\tilde q|=|pq|$, $|\tilde p\tilde r|=|pr|$ and $|\tilde q\tilde r|=|qr|$[†], and then we denote by $\tilde\angle_k qpr$ the angle at $\tilde p$ of the triangle $\triangle\tilde p\tilde q\tilde r$.

Definition 1.1 ([6]). *In an $X\in Alex(k)$, a closed subset F is called to be quasi-convex if the following condition is satisfied: if the distance function to $q\notin F$ restricted to F, $\mathrm{dist}_q|_F$, attains a minimum at $p\in F$, then for all $r\in F\setminus\{p\}$*

$$|\Uparrow_p^q\Uparrow_p^r|\leqslant\frac{\pi}{2}\ \ (\textit{or equivalently, } \tilde\angle_k qpr\leqslant\frac{\pi}{2}).\tag{1.1}$$

Here, we make a convention that both the empty set and a single point are quasi-convex in X.

By Toponogov's Theorem[‡], it is obvious that '$|\Uparrow_p^q\Uparrow_p^r|\leqslant\frac{\pi}{2}$' implies '$\tilde\angle_k qpr\leqslant\frac{\pi}{2}$', but not vice versa; however, they are equivalent to each other in the situation of Definition 1.1 [6]. Moreover, due to the arbitrariness of q, (1.1) is in fact equivalent to $|\uparrow_p^q\Uparrow_p^r|\leqslant\frac{\pi}{2}$ for any $[pq]$.

Remark 1.1. In this paper, we say that F is *extremal* in X if (1.1) holds for all $r\in X\setminus\{p\}$ in Definition 1.1. This coincides with the concept 'extremal' in [3] if F contains at least two points [6]. If F is the empty set or a single point and if $k=1$, some extra conditions are added in [3] for some kind of completeness ([6]).

Remark 1.2. In [6], locally quasi-convex subsets are also defined. In detail, a subset F in an $X\in\mathrm{Alex}(k)$ is *locally quasi-convex* if for any $x\in F$ there is a neighborhood U_x of x such that $U_x\cap F$ is closed and if $\mathrm{dist}_q|_F$ with $q\notin F$ attains a minimum at $p\in F\cap U_x$, then the corresponding (1.1) holds for all $r\in F\cap U_x\setminus\{p\}$. In a complete Riemannian manifold, a closed and locally quasi-convex subset must be a totally geodesic submanifold.

It is obvious that the quasi-convexity (as well as the extremality) of F is determined by the geometry at points realizing minimums of $\mathrm{dist}_q|_F$ with $q\notin F$. A natural question is what is the essential geometry to a general point of F. The first result of this paper gives an answer to it.

[†]If $k>0$ and $|pq|=\frac{\pi}{\sqrt{k}}$, it is necessary to add a condition that $\tilde r$ lies in a geodesic $[\tilde p\tilde q]$.

[‡]For the theorem, one can refer to Section 3 in [2] (or Theorem 1.1 in [6]).