

Quasi-Convex Subsets and the Farthest Direction in Alexandrov Spaces with Lower Curvature Bound

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Abstract. Let F be a closed subset in a finite dimensional Alexandrov space X with lower curvature bound. This paper shows that F is quasi-convex if and only if, for any two distinct points $p, r \in F$, if there is a direction at p which is more than $\frac{\pi}{2}$ away from $\uparrow_p^r$ (the set of all directions from p to r), then the farthest direction to $\uparrow_p^r$ at p is tangent to F . This implies that F is quasi-convex if and only if the gradient curve starting from r of the distance function to p lies in F . As an application, we obtain that the fixed point set of an isometry on X is quasi-convex.

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1 Introduction

Finite dimensional Alexandrov spaces with lower curvature bound can be viewed as a generalization of Riemannian manifolds with lower sectional curvature bound [2]. Compared to Riemannian manifolds, Alexandrov spaces might have some singularities, so that some important subsets appear as some kinds of analogues of totally geodesic submanifolds, such as convex subsets, extremal subsets and quasi-geodesics [3,4]. Recently, such a kind of subsets named quasi-convex subsets has been introduced [6]. They include not only all convex subsets without boundary and extremal subsets but also more other subsets, such as fixed point sets of isometries on Alexandrov spaces. Moreover, all shortest pathes in a quasi-convex subset are quasi-geodesics.

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To show the definition of quasi-convex subsets, we first make some conventions on notations.

- $\text{Alex}(k)$: the set of complete and finite dimensional Alexandrov spaces with curvature $\geq k$.
- S_k^2 : the complete and simply connected space form of dimension 2 and curvature k .
- $|pq|, [pq]$: the distance, a minimal geodesic (i.e. shortest path) between p and q .
- $\uparrow_p^q$: the direction from p to q for a given $[pq]$.
- $\uparrow_p^q$: the set of all directions from p to q .

For a point $p \in X \in \text{Alex}(k)$, we denote by $\Sigma_p X$ the space of directions of X at p which belongs to $\text{Alex}(1)$ [2]. In $\Sigma_p X$, $\uparrow_p^q$ is a closed subset, and $|\uparrow_p^q \uparrow_p^r|$ is the distance between $\uparrow_p^q$ and $\uparrow_p^r$. And to $p, q, r \in X$, we associate $\tilde{p}, \tilde{q}, \tilde{r} \in S_k^2$ with $|\tilde{p}\tilde{q}| = |pq|$, $|\tilde{p}\tilde{r}| = |pr|$ and $|\tilde{q}\tilde{r}| = |qr|^\dagger$, and then we denote by $\tilde{Z}_k qpr$ the angle at $\tilde{p}$ of the triangle $\triangle \tilde{p}\tilde{q}\tilde{r}$.

Definition 1.1 ([6]). In an $X \in \text{Alex}(k)$, a closed subset F is called to be quasi-convex if the following condition is satisfied: if the distance function to $q \notin F$ restricted to F , $\text{dist}_q|_F$, attains a minimum at $p \in F$, then for all $r \in F \setminus \{p\}$

$$|\uparrow_p^q \uparrow_p^r| \leq \frac{\pi}{2} \quad (\text{or equivalently, } \tilde{Z}_k qpr \leq \frac{\pi}{2}). \quad (1.1)$$

Here, we make a convention that both the empty set and a single point are quasi-convex in X .

By Toponogov's Theorem[†], it is obvious that ' $|\uparrow_p^q \uparrow_p^r| \leq \frac{\pi}{2}$ ' implies ' $\tilde{Z}_k qpr \leq \frac{\pi}{2}$ ', but not vice versa; however, they are equivalent to each other in the situation of Definition 1.1 [6]. Moreover, due to the arbitrariness of q , (1.1) is in fact equivalent to $|\uparrow_p^q \uparrow_p^r| \leq \frac{\pi}{2}$ for any $[pq]$.

Remark 1.1. In this paper, we say that F is *extremal* in X if (1.1) holds for all $r \in X \setminus \{p\}$ in Definition 1.1. This coincides with the concept 'extremal' in [3] if F contains at least two points [6]. If F is the empty set or a single point and if $k = 1$, some extra conditions are added in [3] for some kind of completeness ([6]).

Remark 1.2. In [6], locally quasi-convex subsets are also defined. In detail, a subset F in an $X \in \text{Alex}(k)$ is *locally quasi-convex* if for any $x \in F$ there is a neighborhood U_x of x such that $U_x \cap F$ is closed and if $\text{dist}_q|_F$ with $q \notin F$ attains a minimum at $p \in F \cap U_x$, then the corresponding (1.1) holds for all $r \in F \cap U_x \setminus \{p\}$. In a complete Riemannian manifold, a closed and locally quasi-convex subset must be a totally geodesic submanifold.

It is obvious that the quasi-convexity (as well as the extremality) of F is determined by the geometry at points realizing minimums of $\text{dist}_q|_F$ with $q \notin F$. A natural question is what is the essential geometry to a general point of F . The first result of this paper gives an answer to it.

[†]If $k > 0$ and $|pq| = \frac{\pi}{\sqrt{k}}$, it is necessary to add a condition that $\tilde{r}$ lies in a geodesic $[\tilde{p}\tilde{q}]$.

[‡]For the theorem, one can refer to Section 3 in [2] (or Theorem 1.1 in [6]).

Theorem 1.1. *Let F be a closed subset in an $X \in \text{Alex}(k)$. Then F is quasi-convex in X if and only if, for any two distinct points $p, r \in F$ and $\eta \in \Sigma_p X$, there is $\zeta \in \uparrow_p^r$ and $\xi \in \Sigma_p F$ with $|\eta \xi| \leq \frac{\pi}{2}$ such that*

$$\cos|\eta \zeta| \geq \cos|\eta \xi| \cos|\zeta \xi|; \quad (1.2)$$

where if $\Sigma_p F = \emptyset$ (i.e. p is an isolated point of F), then (1.2) means that $\cos|\eta \zeta| \geq 0$.

Note that, in Theorem 1.1, if η lies in $\Sigma_p F$, then we can let $\xi = \eta$. And an alternative formulation of (1.2) is $\angle_1 \eta \xi \leq \frac{\pi}{2}$.

Remark 1.3. In Theorem 1.1, according to the proof of Theorem 1.1, if F is quasi-convex in X , we can in fact select ξ with $|\eta \xi| \leq \frac{\pi}{2}$ to satisfy a bit stronger version of (1.2):

$$\cos|\eta \uparrow_p^r| \geq \cos|\eta \xi| \cos|\uparrow_p^r \xi|.$$

Restricted to ‘extremal’ case, Theorem 1.1 can be formulated as follows, which can be seen almost obviously from the proof of Theorem 1.1.

Corollary 1.1. *Let F be a closed subset in an $X \in \text{Alex}(k)$. Then F is extremal in X if and only if, for any $p \in F$ and any $\eta, \zeta \in \Sigma_p X$, there is $\xi \in \Sigma_p F$ with $|\eta \xi| \leq \frac{\pi}{2}$ such that $\cos|\eta \zeta| \geq \cos|\eta \xi| \cos|\zeta \xi|$ (which means that $\cos|\eta \zeta| \geq 0$ if $\Sigma_p F = \emptyset$).*

From Theorem 1.1, we can derive another equivalent condition of quasi-convexity, which will be very helpful to see some nice properties of quasi-convex subsets.

Corollary 1.2. *Let F be a closed subset in an $X \in \text{Alex}(k)$. Then F is quasi-convex in X if and only if, for any two distinct points $p, r \in F$, if there is $\eta \in \Sigma_p X$ satisfying $|\eta \uparrow_p^r| > \frac{\pi}{2}$, then the farthest direction to $\uparrow_p^r$ in $\Sigma_p X$ belongs to $\Sigma_p F$.*

Note that, the farthest direction ξ to $\uparrow_p^r$ in $\Sigma_p X$, i.e. $|\xi \uparrow_p^r| = \max\{|\nu \uparrow_p^r| : \nu \in \Sigma_p X\}$, is unique if the maximum is bigger than $\frac{\pi}{2}$ (by Toponogov’s Theorem). In Corollary 1.2, if $\Sigma_p F = \emptyset$, then “if there is $\eta \in \Sigma_p X$ satisfying $|\eta \uparrow_p^r| > \frac{\pi}{2}$, then ...” means that $|\eta \uparrow_p^r| \leq \frac{\pi}{2}$ for all $\eta \in \Sigma_p X$. In Sections 3 and 5, we will present two equivalent versions of Corollary 1.2 (see Propositions 3.1 and 5.1).

Similarly, for ‘extremal’ case, Corollary 1.2 can be formulated as follows: In an $X \in \text{Alex}(k)$, a closed subset F is extremal if and only if, for any $p \in F$ and $\zeta \in \Sigma_p X$, if there is $\eta \in \Sigma_p X$ satisfying $|\eta \zeta| > \frac{\pi}{2}$, then the farthest direction to ζ in $\Sigma_p X$ belongs to $\Sigma_p F$. The necessity of the extremality of F here can be seen from [3], and the sufficiency has been given in [5].

Remark 1.4. Based on Remark 1.2, one can give the corresponding versions of Theorem 1.1 and Corollary 1.2 (and Propositions 3.1 and 5.1) for local quasi-convexity.

Remark 1.5. Let F be a quasi-convex subset in $X \in \text{Alex}(k)$. [6] has shown two important properties of F . One is that $\Sigma_p F$ is also quasi-convex in $\Sigma_p X$ for any $p \in F$, and the other is that a shortest path in F is a quasi-geodesic in X . They can be proven just from Definition 1.1 without involving the new viewpoints in Theorem 1.1 and Corollary 1.2.

In the rest of the paper, we first give proofs of Theorem 1.1 and Corollary 1.2 in Sections 2 and 3. Then, as applications of them, we will show several properties of quasi-convex subsets. For instance, two points in a quasi-convex subset can be jointed with a curve in the subset if they are close sufficiently (see Section 4), and the intersection of two quasi-convex subsets is also quasi-convex (see Section 5.2). Moreover, based on Corollary 1.2, we can illustrate quasi-convex subsets by gradient curves of distance functions (see Section 5.1). This will make it easy to judge some kinds of subsets to be quasi-convex (see Sections 5.2-5.4), such as the fixed point set of an isometry.

2 Proofs of Theorem 1.1 and Corollary 1.1

The main goal of this section is to give a proof for Theorem 1.1. In the proof, we will use the following facts in Alexandrov geometry.

Lemma 2.1 ([2]). *Let $X \in \text{Alex}(k)$ and $p \in X$. Then for any small $\epsilon > 0$ there is a neighborhood U_ϵ of p such that, for any $[pq], [pr], [qr] \subset U_\epsilon$,*

$$0 \leq |\uparrow_p^q \uparrow_p^r| - \tilde{Z}_k qpr < \epsilon, \quad 0 \leq |\uparrow_q^p \uparrow_q^r| - \tilde{Z}_k pqr < \epsilon \quad \text{and} \quad 0 \leq |\uparrow_r^p \uparrow_r^q| - \tilde{Z}_k prq < \epsilon.$$

Lemma 2.2 (The first variation formula [2]). *Let $X \in \text{Alex}(k)$, and let $p, r, q_i \in X$ with $q_i \rightarrow p$ as $i \rightarrow \infty$. If there is $[pq_i]$ such that $\uparrow_p^{q_i}$ converges to η ($\in \Sigma_p X$) as $i \rightarrow \infty$, then*

$$\lim_{i \rightarrow \infty} \frac{|q_i r| - |pr|}{|pq_i|} = -\cos |\uparrow_p^r \eta|.$$

As a result, $\lim_{i \rightarrow \infty} \tilde{Z}_k r p q_i = |\uparrow_p^r \eta|$.

Proof of Theorem 1.1. We first show the necessity of the quasi-convexity in the theorem. Note that we can let $\xi = \eta$ if $\eta \in \Sigma_p F$, so we can assume that $\eta \notin \Sigma_p F$. Then there is $q_n \in X \setminus F$ such that $q_n \rightarrow p$ and $\uparrow_p^{q_n} \rightarrow \eta$ as $n \rightarrow \infty$. Let $p_n \in F$ satisfy $|q_n p_n| = |q_n F|$. Note that $p_n \rightarrow p$ as $n \rightarrow \infty$, and we can complete the proof according to the following two cases.

Case 1: there is at least a subsequence of $\{p_n\}$ which belongs to $F \setminus \{p\}$. In this case, there must be a subsequence $\{p_{n_i}\}$ with $p_{n_i} \neq p$ such that $\uparrow_p^{p_{n_i}}$ converges to some $\xi \in \Sigma_p F$ as $i \rightarrow \infty$. We claim that $|\eta \xi| \leq \frac{\pi}{2}$ and

$$\cos |\eta \uparrow_p^r| \geq \cos |\eta \xi| \cos |\uparrow_p^r \xi|. \quad (2.1)$$

This implies that there is $\zeta \in \uparrow_p^r$ such that (1.2) holds. So, we just need to verify the claim in this case. For simpleness, we still denote by $\{p_n\}$ the subsequence $\{p_{n_i}\}$. Since $|q_n p_n| < |q_n p|$ and $q_n, p_n \rightarrow p$ as $n \rightarrow \infty$, by Lemma 2.1 it is easy to see that $|\eta \xi| \leq \frac{\pi}{2}$. For (2.1), due to the similarity, we only give a proof for the case where $k=0$. By the quasi-convexity of F , we have that

$$\tilde{Z}_k q_n p_n r \leq \frac{\pi}{2}. \quad (2.2)$$

Consequently, by the Law of Cosine, we have that

$$|rp_n|^2 + |q_np_n|^2 \geq |q_nr|^2,$$

where

$$\begin{aligned} |rp_n|^2 &= |rp|^2 + |pp_n|^2 - 2|rp| \cdot |pp_n| \cdot \cos \tilde{Z}_0 rpp_n, \\ |q_np_n|^2 &= |q_np|^2 + |pp_n|^2 - 2|q_np| \cdot |pp_n| \cdot \cos \tilde{Z}_0 q_npp_n, \\ |q_nr|^2 &= |q_np|^2 + |pr|^2 - 2|q_np| \cdot |pr| \cdot \cos \tilde{Z}_0 q_npr. \end{aligned}$$

It then follows that

$$\cos \tilde{Z}_0 rpq_n \geq -\frac{|pp_n|^2}{|pq_n| \cdot |pr|} + \frac{|pp_n|}{|pq_n|} \cos \tilde{Z}_0 rpp_n + \frac{|pp_n|}{|pr|} \cos \tilde{Z}_0 q_npp_n. \quad (2.3)$$

Since $q_n, p_n \rightarrow p$, $\uparrow_p^{q_n} \rightarrow \eta$ and $\uparrow_p^{p_n} \rightarrow \xi$ as $n \rightarrow \infty$, by Lemma 2.2 we have that

$$\tilde{Z}_0 rpq_n \rightarrow |\uparrow_p^r \eta| \quad \text{and} \quad \tilde{Z}_0 rpp_n \rightarrow |\uparrow_p^r \xi| \quad \text{as } n \rightarrow \infty. \quad (2.4)$$

Moreover, we make a subclaim:

$$\lim_{n \rightarrow \infty} \frac{|pp_n|}{|pq_n|} = \cos |\eta \xi|. \quad (2.5)$$

Note that (2.3)-(2.5) together with $\frac{|pp_n|}{|pr|} \rightarrow 0$ as $n \rightarrow \infty$ implies (2.1). In order to see the subclaim, by Lemma 2.1 it suffices to show that there is $[p_nq_n]$ and $[p_np]$ such that $|\uparrow_{p_n}^{q_n} \uparrow_{p_n}^p| \rightarrow \frac{\pi}{2}$ as $n \rightarrow \infty$. By the quasi-convexity of F , there is $[p_nq_n]$, $[p_np]$ and $[p_np_m]$ with $m \neq n$ such that

$$|\uparrow_{p_n}^{q_n} \uparrow_{p_n}^p| \leq \frac{\pi}{2} \quad \text{and} \quad |\uparrow_{p_n}^{q_n} \uparrow_{p_n}^{p_m}| \leq \frac{\pi}{2}.$$

On the other hand, since $p_n \rightarrow p$ and $\uparrow_p^{p_n} \rightarrow \xi$ as $n \rightarrow \infty$, it is not hard to see that $|\uparrow_{p_n}^p \uparrow_{p_n}^{p_m}| \rightarrow \pi$ where $n \gg m$ and $m \rightarrow \infty$ (by Toponogov's Theorem), which implies

$$|\uparrow_{p_n}^{q_n} \uparrow_{p_n}^p| + |\uparrow_{p_n}^{q_n} \uparrow_{p_n}^{p_m}| \rightarrow \pi.$$

It therefore follows that $|\uparrow_{p_n}^{q_n} \uparrow_{p_n}^p| \rightarrow \frac{\pi}{2}$ as $n \rightarrow \infty$ (i.e. the subclaim is verified, so is the claim).

Case 2: $p_n = p$ for all large n . In this case, the quasi-convexity of F guarantees that $|\uparrow_p^{q_n} \uparrow_p^r| \leq \frac{\pi}{2}$, and thus $|\eta \uparrow_p^r| \leq \frac{\pi}{2}$ (note that ' $q_n \rightarrow p$ and $\uparrow_p^{q_n} \rightarrow \eta$ ' implies $\uparrow_p^{q_n} \rightarrow \eta$, [2]). I.e., there is $\zeta \in \uparrow_p^r$ such that $\cos |\eta \zeta| \geq 0$, so the proof of the necessity is done if $\Sigma_p F = \emptyset$. If $\Sigma_p F \neq \emptyset$, we claim that

$$|\uparrow_p^{q_n} \zeta| = \frac{\pi}{2} \quad \text{for any } \zeta \in \Sigma_p F,$$

which implies $|\eta\zeta| = \frac{\pi}{2}$. In fact, we have that $|\uparrow_p^{q_n}\zeta| \geq \frac{\pi}{2}$ because $|q_n p| = |q_n F|$ (by Lemma 2.2), and meanwhile $|\uparrow_p^{q_n}\zeta| \leq \frac{\pi}{2}$ (by the quasi-convexity of F). Note that the claim right above implies that any $\zeta \in \Sigma_p F$ satisfies (1.2), and thus the proof of the necessity is done.

Next, we will verify the sufficiency of the quasi-convexity in Theorem 1.1. We argue by contradiction. Suppose that F is not quasi-convex in X . Then there exists $q \notin F$ and $p, r \in F$ such that $|qp| = |qF|$, but for some $[qp]$ we have that

$$|\uparrow_p^q \uparrow_p^r| > \frac{\pi}{2}. \quad (2.6)$$

We let $\eta \triangleq \uparrow_p^q$. By the assumption, there is $\zeta \in \uparrow_p^r$ and $\xi \in \Sigma_p F$ with $|\eta\xi| \leq \frac{\pi}{2}$ such that

$$\cos|\eta\zeta| \geq \cos|\eta\xi| \cdot \cos|\zeta\xi|.$$

Note that $'|\eta\zeta| > \frac{\pi}{2}'$ (see (2.6)) together with $'|\eta\xi| \leq \frac{\pi}{2}'$ implies that $|\eta\xi|$ must be less than $\frac{\pi}{2}$. However, since $|qp| = |qF|$, by Lemma 2.2 we have that $|\eta\xi| \geq \frac{\pi}{2}$, a contradiction. $\square$

By restricting the above proof to 'extremal' case, we can easily derive Corollary 1.1.

Proof of Corollary 1.1. If ζ is the direction of some $[pr]$ at p (here r is not needed to lie in F), one just need to replace all 'quasi-convexity' with 'extremality' in the proof of Theorem 1.1. Then for the case where ζ cannot be realized by a minimal geodesic, we can draw the conclusion by a limiting argument (note that there is $[pr_i]$ with $r_i \rightarrow p$ as $i \rightarrow \infty$ such that $\uparrow_p^{r_i} \rightarrow \zeta$). $\square$

Remark 2.1. Similar to Corollary 1.1, if F is a nonempty quasi-convex subset in $X \in \text{Alex}(k)$, and if $\Sigma_p F \neq \emptyset$ with $p \in F$, then for any $\eta \in \Sigma_p X$ and $\zeta \in \Sigma_p F$ there is $\xi \in \Sigma_p F$ with $|\eta\xi| \leq \frac{\pi}{2}$ such that $\cos|\eta\zeta| \geq \cos|\eta\xi| \cos|\zeta\xi|$ (by Theorem 1.1). However, the converse might not be true (as an example, one can consider a submanifold in a Riemannian manifold which is not totally geodesic).

3 Proof of Corollary 1.2

In this section, we will first prove Corollary 1.2, and then show an alternative version of it.

Proof of Corollary 1.2. We first verify the necessity of the quasi-convexity in the corollary. Let $p, r \in F$. Since there is $\eta \in \Sigma_p X$ such that $|\eta \uparrow_p^r| > \frac{\pi}{2}$, the farthest direction ζ_0 to $\uparrow_p^r$ in $\Sigma_p X$ is unique and $|\zeta_0 \uparrow_p^r| > \frac{\pi}{2}$. By Theorem 1.1 (and Remark 1.3, see (2.1)), the quasi-convexity of F implies that there is $\xi \in \Sigma_p F$ with $|\zeta_0 \xi| \leq \frac{\pi}{2}$ such that

$$\cos|\zeta_0 \uparrow_p^r| \geq \cos|\zeta_0 \xi| \cos|\uparrow_p^r \xi|.$$

Note that $'|\zeta_0 \uparrow_p^r| > \frac{\pi}{2}'$, $'|\zeta_0 \uparrow_p^r| \geq |\zeta \uparrow_p^r|'$ and $'|\zeta_0 \xi| \leq \frac{\pi}{2}'$ imply that ζ_0 has to be equal to ξ , so it follows that $\zeta_0 \in \Sigma_p F$.

Next, we will verify the sufficiency, and argue by contradiction. Suppose that F is not quasi-convex in X . Then there exists $q \notin F$ and $p, r \in F$ such that $|qp| = |qF|$, but for some $[qp]$ we have that

$$|\uparrow_p^r \uparrow_p^q| > \frac{\pi}{2}.$$

Hence, by the assumption, the farthest direction ξ to $\uparrow_p^r$ in $\Sigma_p X$ belongs to $\Sigma_p F$. We claim that $|\uparrow_p^q \xi| < \frac{\pi}{2}$, which contradicts ' $|qp| = |qF|$ ' (by Lemma 2.2). In fact, if $|\uparrow_p^q \xi| \geq \frac{\pi}{2}$, by Toponogov's Theorem it is not hard to see that $|\uparrow_p^r \xi'| > |\uparrow_p^r \xi|$ for ξ' near ξ in any $[\uparrow_p^q \xi]$ because $|\uparrow_p^r \uparrow_p^q| > \frac{\pi}{2}$ and $|\uparrow_p^r \xi| \geq |\uparrow_p^r \uparrow_p^q| > \frac{\pi}{2}$; a contradiction (because ξ is the farthest direction to $\uparrow_p^r$). $\square$

We now provide an alternative version of Corollary 1.2, which will be used in next section. We formulate it in the following proposition.

Proposition 3.1. *Let F be a closed subset in an $X \in \text{Alex}(k)$. Then F is quasi-convex in X if and only if, for any two distinct points $p, r \in F$, if $\eta \in \Sigma_p X$ satisfies $-\cos|\uparrow_p^r \eta| > 0$, then there exists $\xi \in \Sigma_p F$ such that*

$$-\cos|\uparrow_p^r \xi| \geq -\cos|\uparrow_p^r \eta|. \quad (3.1)$$

Moreover, when F is quasi-convex and $-\cos|\uparrow_p^r \eta| > 0$, the ξ in (3.1) can be chosen to satisfy $\cos|\eta \xi| \geq -\cos|\uparrow_p^r \eta|$; as a result, there is $\xi' \in \Sigma_p F$ such that

$$\cos|\uparrow_p^r \xi'| \geq -\cos|\uparrow_p^r \eta|. \quad (3.2)$$

Proof. Note that the first statement is an alternative formulation of Corollary 1.2. So, we just need to verify the second one. Assume that F is quasi-convex and $\eta \in \Sigma_p X$ satisfies $-\cos|\uparrow_p^r \eta| > 0$. By Theorem 1.1, there is $\zeta \in \uparrow_p^r$ and $\xi \in \Sigma_p F$ with $|\eta \xi| \leq \frac{\pi}{2}$ such that

$$\cos|\eta \xi| \geq \cos|\eta \zeta| \cos|\zeta \xi|.$$

This implies that $\cos|\eta \xi| \geq -\cos|\uparrow_p^r \eta|$ because $-\cos|\eta \zeta| \geq -\cos|\uparrow_p^r \eta| > 0$ and $\cos|\eta \xi| \geq 0$.

We next show that there is $\xi' \in \Sigma_p F$ such that (3.2) holds. Note that there is a sequence of $r_i \in F$ such that $r_i \rightarrow p$ and $\uparrow_p^{r_i} \rightarrow \zeta$ as $i \rightarrow \infty$. And for any $\zeta \in \uparrow_p^r$, by replacing $\uparrow_p^r$ and η with $\uparrow_p^{r_i}$ and ζ respectively, we can conclude that there is $\xi_i \in \Sigma_p F$ such that $\cos|\zeta \xi_i| \geq -\cos|\uparrow_p^{r_i} \zeta|$, which implies $\cos|\uparrow_p^r \xi_i| \geq -\cos|\uparrow_p^{r_i} \zeta|$. Thereby, for the limit ξ' of any converging subsequence of $\{\xi_i\}$ (note that $\Sigma_p F$ is a closed subset in $\Sigma_p X$), we have that $\cos|\uparrow_p^r \xi'| \geq -\cos|\uparrow_p^r \zeta| \geq -\cos|\uparrow_p^r \eta|$. $\square$

Remark 3.1. By replacing $\uparrow_p^r$ with any $\zeta \in \Sigma_p X$ in Proposition 3.1, we can get the corresponding version of the proposition for 'extremal' case. Here, the existences of ξ and ξ' by the extremality of F can be seen from Proposition 1.6 in [3].

In the rest of the paper, we will present some properties of quasi-convex subsets, as applications of the idea of Theorem 1.1 (and its equivalent versions—Corollary 1.2 and Proposition 3.1).

4 Connectedness of quasi-convex subsets

It is known that the number of extremal subsets in a compact space of $\text{Alex}(k)$ is finite (Proposition 3.6 in [3]). Unfortunately, there is no such strong conclusion on quasi-convex subsets. For example, in a standard sphere, there are infinitely many great circles each of which is quasi-convex. Nevertheless, we have a weaker conclusion for quasi-convex subsets.

Proposition 4.1. *In a compact space $X \in \text{Alex}(k)$, any quasi-convex subset has a finite number of connected components.*

In fact, we have the following stronger conclusion than Proposition 4.1 (which corresponds to (2) of Corollary 3.2 in [3] for ‘extremal’ case).

Proposition 4.2. *Let X be a compact space in $\text{Alex}(k)$, and let F be a quasi-convex subset in X . Then there is $\epsilon > 0$ (depending on X) such that, for any two distinct points $p, q \in F$ with $|pq| < \epsilon^2$, there exists a curve in F joining p and q with length bounded from above by $\frac{|pq|}{\epsilon}$.*

Moreover, we can see the following property which corresponds to Proposition 3.3 in [3] for ‘extremal’ case.

Proposition 4.3. *Let F be a quasi-convex subset in an $X \in \text{Alex}(k)$, and let $p \in F$ and $\xi \in \Sigma_p F$. Then there exists a curve in F starting from p and tangent to the direction ξ .*

In proving Propositions 4.1 and 4.2, the following lemma (Lemma 3.1 in [3]) is needed, which is some kind of essential geometry of Alexandrov spaces with lower curvature bound.

Lemma 4.1. *Let X be a compact space in $\text{Alex}(k)$. Then there is $\epsilon > 0$ (depending on X) such that, for any two distinct points $p, q \in X$ with $|pq| < \epsilon^2$, at least one of the following holds:*

$$\max_{\eta \in \Sigma_q X} \text{dist}'_p|_q(\eta) > \epsilon \quad \text{and} \quad \max_{\eta \in \Sigma_p X} \text{dist}'_q|_p(\eta) > \epsilon,$$

where $\text{dist}'_p|_q(\eta)$ denotes the derivative of dist_p (the distance function to p) at q along the direction η .

Note that $\text{dist}'_p|_q(\eta) = -\cos|\uparrow_q^p \eta|$ (by Lemma 2.2). Hence, by Proposition 3.1 (see (3.1) and (3.2)) we can easily see the following property.

Lemma 4.2. *In Lemma 4.1, if p, q lie in a quasi-convex subset F of X additionally, then*

$$\max_{\xi \in \Sigma_q F} \text{dist}'_p|_q(\xi) > \epsilon \quad \text{and} \quad \min_{\xi \in \Sigma_p F} \text{dist}'_q|_p(\xi) < -\epsilon,$$

or

$$\max_{\xi \in \Sigma_p F} \text{dist}'_q|_p(\xi) > \epsilon \quad \text{and} \quad \min_{\xi \in \Sigma_q F} \text{dist}'_p|_q(\xi) < -\epsilon.$$

Lemma 4.2 corresponds to (1) of Corollary 3.2 in [3] for ‘extremal’ case, where p and q can lie in two distinct extremal subsets.

Since our proofs for Propositions 4.1 and 4.2 are imitations of their corresponding versions for ‘extremal’ case in [3], we just provide rough proofs for them.

Proof of Proposition 4.1. Let F be a quasi-convex subset in X , and let F_1 and F_2 be two connected components of F . It suffices to show that the distance between F_1 and F_2 is bigger than ϵ^2 , where ϵ is the number associated to X satisfying Lemma 4.1. If this is not true, then there is $p \in F_1$ and $q \in F_2$ such that $|pq| = |F_1 F_2| < \epsilon^2$ (note that X is compact and F is closed in X). It then follows that

$$\min_{\xi \in \Sigma_p F_1} \text{dist}'_q|_p(\xi) \geq 0 \quad \text{and} \quad \min_{\xi \in \Sigma_q F_2} \text{dist}'_p|_q(\xi) \geq 0,$$

which contradicts Lemma 4.2. $\square$

Proof of Proposition 4.2. Let p, q be two distinct points in F . By Lemma 4.2, there is an ϵ such that if $|pq| < \epsilon^2$, then

$$\min_{\xi \in \Sigma_q F} \text{dist}'_p|_q(\xi) < -\epsilon \quad \text{or} \quad \min_{\xi \in \Sigma_p F} \text{dist}'_q|_p(\xi) < -\epsilon.$$

Then, for each $n \in \mathbb{N}^+$, it is not hard to see that there are two sequences of points $\{p_i\}_{i=1}^\infty$ and $\{q_i\}_{i=1}^\infty$ (depending on n) in F with

$$\text{either } p_i = p_{i-1} \quad \text{and} \quad 0 < |q_i q_{i-1}| < \frac{1}{n} \quad \text{or} \quad q_i = q_{i-1} \quad \text{and} \quad 0 < |p_i p_{i-1}| < \frac{1}{n} \quad (4.1)$$

(where $p_0 = p$ and $q_0 = q$) such that

$$|p_{i-1} q_{i-1}| - |p_i q_i| > \epsilon(|p_{i-1} p_i| + |q_i q_{i-1}|) \quad (4.2)$$

and

$$|p_i q_i| \rightarrow 0 \text{ as } i \rightarrow \infty. \quad (4.3)$$

Let $\{p_{i,n}, q_{i,n}\}_{i=0}^\infty$ denote the above two point sequences corresponding to each n . Note that

$$\sum_{i=1}^\infty (|p_{i-1,n} p_{i,n}| + |q_{i,n} q_{i-1,n}|) < \frac{|pq|}{\epsilon}$$

for all n . Hence, as $n \rightarrow \infty$ and passing to a subsequence of $\{n\}$, $\{p_{i,n}, q_{i,n}\}_{i=0}^\infty$ converges to a curve in F joining p and q with length $\leq \frac{|pq|}{\epsilon}$. $\square$

Remark 4.1. In the proof of Proposition 4.2, for general $\{p_i\}_{i=1}^\infty$ and $\{q_i\}_{i=1}^\infty$ satisfying (4.1) and (4.2), it might occur that $p_i \rightarrow \bar{p}$ and $q_i \rightarrow \bar{q}$ as $i \rightarrow \infty$ with $\bar{p} \neq \bar{q}$. Note that, $|\bar{p}\bar{q}| < |pq| < \epsilon^2$, so similarly there is p' and q' in F with either $p' = \bar{p}$ and $|q'\bar{q}| < \frac{1}{n}$ or $q' = \bar{q}$ and $|p'\bar{p}| < \frac{1}{n}$ such that $|\bar{p}\bar{q}| - |p'q'| > \epsilon(|\bar{p}p'| + |q'\bar{q}|)$. Hence, for sufficiently large i , we can reset $p_i = p_{i-1}$ and $q_i = q'$ or $q_i = q_{i-1}$ and $p_i = p'$ so that the new p_i and q_i still satisfy (4.1) and (4.2). Such an idea enables us to find $\{p_i\}_{i=1}^\infty$ and $\{q_i\}_{i=1}^\infty$ satisfying (4.1)-(4.3).

As for Proposition 4.3, we can almost directly copy the proof of Proposition 3.3 in [3] (here the basis is Lemma 4.2 instead of Corollary 3.2 in [3]). However, we would like to provide a proof for it via Corollary 1.2 without involving Lemmas 4.2 and 4.1.

Proof of Proposition 4.3. Since $\xi \in \Sigma_p F$, there is $\{p_i\}_{i=1}^\infty \subset F$ such that $p_i \rightarrow p$ and $\uparrow_p^{p_i} \rightarrow \xi$ as $i \rightarrow \infty$. Note that, for sufficiently small $\delta > 0$, there is i_0 such that

$$|\uparrow_p^{p_i} \uparrow_p^{p_{i_0}}| < \frac{\delta}{3} \quad \text{for all } i > i_0. \quad (4.4)$$

Moreover, by Lemma 2.1, we can assume that

$$|\uparrow_p^x \uparrow_p^y| - \tilde{Z}_k x p y < \frac{\delta}{3} \quad \text{for any } [px], [py] \subset \overline{B_p(|pp_{i_0}|)}. \quad (4.5)$$

In the rest of the proof, for any $i > i_0$ with $|pp_i| \ll |pp_{i_0}|$, we will first construct an arc-length parameterized curve $\alpha_i(t) \big|_{t \in [0, \frac{|p_i p_{i_0}|}{2}]}$ $\subset F$ with $\alpha_i(0) = p_i$ such that

$$|\uparrow_p^{\alpha_i(t)} \xi| < \delta \quad \text{for all } t \in [0, \frac{|p_i p_{i_0}|}{2}].$$

And then we will show that, as $i \rightarrow \infty$, $\alpha_i(t)$ converges to a curve we want.

In order to construct $\alpha_i(t)$, we claim that, for any $\epsilon \ll |pp_i|$, there is $\{z_j\}_{j=1}^{N(\epsilon)} \subset F$ with $z_1 = p_i$ and a constant C such that

$$|z_j z_{j+1}| < \epsilon, \quad \left| \sum_{j=1}^{N(\epsilon)-1} |z_j z_{j+1}| - \frac{|p_i p_{i_0}|}{2} \right| < \epsilon \quad \text{and} \quad \tilde{Z}_k z_{j+1} p p_{i_0} < \tilde{Z}_k z_j p p_{i_0} + \frac{C\epsilon}{|pp_i|} |z_j z_{j+1}|.$$

Note that, (4.4) implies that $\tilde{Z}_k z_1 p p_{i_0} < \frac{\delta}{3}$ (by Toponogov's Theorem) and $|\uparrow_p^{p_{i_0}} \xi| \leq \frac{\delta}{3}$. Then taking into account (4.5), we can see that $\{z_j\}_{j=1}^{N(\epsilon)}$ converges to the desired $\alpha_i(t)$ as $\epsilon \rightarrow 0$.

We now verify the above claim. Since $\tilde{Z}_k p_i p p_{i_0} < \frac{\delta}{3}$ and $\tilde{Z}_k p p_{i_0} p_i \ll \frac{\delta}{3}$ (note that $|pp_i| \ll |p_i p_{i_0}|$), we can assume that $\tilde{Z}_k p p_i p_{i_0} > \pi - \delta$, and thus by Toponogov's Theorem we have that

$$|\uparrow_{p_i}^p \uparrow_{p_i}^{p_{i_0}}| > \pi - \delta.$$

By Corollary 1.2, the farthest direction ξ_i to $\uparrow_{p_i}^p$ in $\Sigma_{p_i} X$ belongs to $\Sigma_{p_i} F$. By Lemma 2.2 and Toponogov's Theorem, the 'farthest' property of ξ_i implies that for any $\eta \in \uparrow_{p_i}^{p_{i_0}}$ there is $\zeta \in \uparrow_{p_i}^p$ such that $\tilde{Z}_1 \zeta \xi_i \eta \leq \frac{\pi}{2}$. It then follows that

$$\cos |\uparrow_{p_i}^p \uparrow_{p_i}^{p_{i_0}}| \geq \cos |\zeta \eta| \geq \cos |\xi_i \zeta| \cdot \cos |\xi_i \eta| \geq \cos |\xi_i \zeta| \cdot \cos |\xi_i \uparrow_{p_i}^{p_{i_0}}|$$

(note that, $|\xi_i \zeta| \geq |\xi_i \uparrow_{p_i}^p| \geq |\uparrow_{p_i}^p \uparrow_{p_i}^{p_{i_0}}| > \pi - \delta$, and thus it holds that $|\xi_i \eta| < \frac{\pi}{2}$), which implies that

$$\cos |\uparrow_{p_i}^p \uparrow_{p_i}^{p_{i_0}}| \geq -\cos |\xi_i \uparrow_{p_i}^{p_{i_0}}|, \text{ or equivalently, } |\uparrow_{p_i}^p \uparrow_{p_i}^{p_{i_0}}| + |\xi_i \uparrow_{p_i}^{p_{i_0}}| \leq \pi. \quad (4.6)$$

Denote by z_1 the point p_i , and note that $\xi_i \in \Sigma_{z_1} F$. Then for any $\epsilon \ll |pz_1|$, (4.6) together with Lemma 2.2 implies that there is $z_2 \in F$ such that $|z_1 z_2| < \epsilon$ and

$$\tilde{Z}_k pz_1 p_{i_0} + \tilde{Z}_k p_{i_0} z_1 z_2 < \pi + \epsilon. \quad (4.7)$$

Let $\bar{z}_2, \bar{p}, \bar{p}_{i_0} \in S_k^2$ satisfy $|\bar{p}\bar{p}_{i_0}| = |pp_{i_0}|$, $|\bar{z}_2\bar{p}_{i_0}| = |z_2 p_{i_0}|$ and $|\bar{p}\bar{z}_2| = |pz_1| + |z_1 z_2|$. For a special case of (4.7) where $\tilde{Z}_k pz_1 p_{i_0} + \tilde{Z}_k p_{i_0} z_1 z_2 \leq \pi$, we notice that $\angle \bar{z}_2 \bar{p} \bar{p}_{i_0} \leq \tilde{Z}_k z_1 p p_{i_0} (< \frac{\delta}{3})$ by Alexandrov's lemma (Lemma 2.5 in [2]), which implies $\angle \bar{p} \bar{z}_2 \bar{p}_{i_0} > \pi - \delta (> \frac{\pi}{2})$. From ' $\angle \bar{p} \bar{z}_2 \bar{p}_{i_0} > \frac{\pi}{2}$ ', it follows that $\tilde{Z}_k z_2 p p_{i_0} \leq \angle \bar{z}_2 \bar{p} \bar{p}_{i_0}$ (note that $|pz_2| \leq |\bar{p}\bar{z}_2|$), so

$$\tilde{Z}_k z_2 p p_{i_0} < \tilde{Z}_k z_1 p p_{i_0} \quad \text{and} \quad \tilde{Z}_k pz_2 p_{i_0} > \pi - \delta. \quad (4.8)$$

In general, it is not so hard to conclude that $\angle \bar{z}_2 \bar{p} \bar{p}_{i_0} < \tilde{Z}_k z_1 p p_{i_0} + \frac{C\epsilon}{|pp_i|} |z_1 z_2|$, where C is a constant depending only on $|pp_{i_0}|$. Then we can similarly see that

$$\tilde{Z}_k z_2 p p_{i_0} \leq \angle \bar{z}_2 \bar{p} \bar{p}_{i_0} < \tilde{Z}_k z_1 p p_{i_0} + \frac{C\epsilon}{|pp_i|} |z_1 z_2| \quad \text{and} \quad \tilde{Z}_k pz_2 p_{i_0} > \pi - \delta. \quad (4.9)$$

Note that $|\uparrow_{z_1}^p \xi_i| > \pi - \delta$, which implies that $|pz_2| > |pz_1|$ (by Lemma 2.2). Then based on (4.8) and (4.9), we can similarly locate z_j with $j \geq 3$ one by one. Moreover, for a similar reason in Remark 4.1, there is an $N(\epsilon)$ such that

$$\left| \sum_{j=1}^{N(\epsilon)-1} |z_j z_{j+1}| - \frac{|p_i p_{i_0}|}{2} \right| < \epsilon.$$

So far, the claim has been verified.

Note that, $\alpha_i(t)$ converges to a curve $\alpha(t)|_{t \in [0, \frac{|pp_{i_0}|}{2}]}$ with $\alpha(0) = p$ as $i \rightarrow \infty$ (here, there might be a need of passing to a subsequence, but in fact not; cf. Remark 4.2 below). We just need to show that $\alpha(t)$ is tangent to ξ at p . In fact, for any integer $n > 1$, there is $i_n \gg i_0$ such that $|\uparrow_p^{p_i} \uparrow_p^{p_{i_n}}| < \frac{\delta^n}{3}$ for all $i > i_n$ (cf. (4.4)). Then we can similarly construct another curve $\bar{\alpha}_i(t)|_{t \in [0, \frac{|p_i p_{i_n}|}{2}]} \subset F$ with $\bar{\alpha}_i(0) = p_i$ such that $|\uparrow_p^{\bar{\alpha}_i(t)} \xi| < \delta^n$ for all $t \in [0, \frac{|p_i p_{i_n}|}{2}]$. Note that z_j can be chosen to ensure that $\bar{\alpha}_i(t)|_{t \in [0, \frac{|p_i p_{i_n}|}{2}]} = \alpha_i(t)|_{t \in [0, \frac{|p_i p_{i_n}|}{2}]}$; namely, for sufficiently large i we have that

$$|\uparrow_p^{\alpha_i(t)} \xi| < \delta^n \quad \text{for all } t \in [0, \frac{|p_i p_{i_n}|}{2}].$$

This implies that $|\uparrow_p^{\alpha(t)} \xi| < \delta^n$ for all $t \in (0, \frac{|pp_{i_n}|}{2}]$, i.e. $\alpha(t)$ is tangent to ξ at p . $\square$

Remark 4.2. We would like to point out that (4.8) and (4.9) are inspired by the proof of Proposition 3.3 in [3]. Moreover, in the proof right above, $\alpha_i(t)$ is in fact the beginning part of the gradient curve of dist_p starting from p_i , and $\alpha(t)$ is just the beginning part of the radial curve starting from p with direction ξ (Corollary 5.1 below).

5 Gradient curve and its applications

5.1 Gradient and radial curves

In [4], and see also [1, 5], semiconcave functions and their gradient curves have been introduced on an $X \in \text{Alex}(k)$. A very important kind of semiconcave functions is distance function. Let $f: \Omega \subset X \rightarrow \mathbb{R}$ be a semiconcave function. The semiconcavity of f implies that, at any $p \in \Omega$, either for any $\xi \in \Sigma_p X$, $f'_p(\xi) \leq 0$ (where $f'_p(\xi)$ denotes the derivative of f at p along the direction ξ), or there exists a unique $\xi_0 \in \Sigma_p X$ such that $f'_p(\xi_0) = \max\{f'(\xi) | \xi \in \Sigma_p X\} > 0$ (for details refer to [5]); then correspondingly, we can define the gradient vector of f at p , denoted by $\nabla_p f$, to be 0 in the former cases, and $\nabla_p f = f'_p(\xi_0)\xi_0$ in the latter cases. Given a point $q \in \Omega$, we consider a locally Lipschitz curve $\alpha: [0, t_0) \rightarrow \Omega \subset X$ with $\alpha(0) = q$ satisfying that the right tangent vector $\alpha^+(t)$ exists for all $t \in [0, t_0)$ and

$$\alpha^+(t) = \nabla_{\alpha(t)} f.$$

A *gradient curve* of f starting from q is such an α where t_0 is the maximum of all possible t_0 and t_0 might be $+\infty$. It is of interest that the gradient curve $\alpha(t)$ of f starting from any point $r \in \Omega$ exists and is unique ([1, 16.15-18]). Here, if $\nabla_r f = 0$, then $\alpha(t)$ will be just the point r . If $\nabla_r f \neq 0$, then there is a $\delta > 0$ such that for any given $\epsilon > 0$ we can select a sequence of points $r = p_0, p_1, \dots, p_{n_\epsilon} \in X$ such that $|\uparrow_{p_i}^{p_{i+1}} \frac{\nabla_{p_i} f}{|\nabla_{p_i} f|}| < \epsilon$, $|p_i p_{i+1}| < \epsilon$ and $\sum_{i=0}^{n_\epsilon-1} |p_i p_{i+1}| \geq \delta$. Letting $\epsilon \rightarrow 0^+$ and passing to a subsequence, one can conclude that the sequence $\{p_i\}_{i=1}^{n_\epsilon}$ converges to an arc-length parameterized curve, and then by reparameterizing it we can obtain the desired gradient curve. From this construction, one can easily conclude that if a closed subset $F \subset X$ with $r \in F$ satisfies that $\frac{\nabla_q f}{|\nabla_q f|} \in \Sigma_q F$ for any $q \in F$ with $\nabla_q f \neq 0$, then we can select the sequence $\{p_i\}_{i=1}^{n_\epsilon}$ in F so that the corresponding gradient curve starting from r falls in F . That is, the following holds.

Theorem 5.1. *Let F be a closed subset in $X \in \text{Alex}(k)$. For a semiconcave function f defined on a subset $\Omega \subset X$, if $\frac{\nabla_q f}{|\nabla_q f|} \in \Sigma_q F$ for any $q \in F$ with $\nabla_q f \neq 0$, then the gradient curve of f starting from any $r \in F \cap \Omega$ lies in F .*

Via Theorem 5.1 and Corollary 1.2, we can see the following important proposition about quasi-convex subsets, in which we also call the gradient curve $\alpha(t)$ with $t \geq 0$ and $\alpha(0) = r$ the *gradient flow* from r , denoted by $\Phi_f^t(r)$.

Proposition 5.1. *Let F be a closed subset in an $X \in \text{Alex}(k)$. Then F is quasi-convex if and only if, for any two distinct points $p, r \in F$, the gradient curve of dist_p starting from r lies in F , or equivalently, the gradient flow $\Phi_{\text{dist}_p}^t(r) \in F$ for all $t \geq 0$.*

Proof. Note that the distance function dist_p on X is semiconcave on $X \setminus \{p\}$ [5]. If F is quasi-convex, then $\nabla_r \text{dist}_p = 0$ if $|\uparrow_r^p \xi| \leq \frac{\pi}{2}$ for all $\xi \in \Sigma_r X$ by Lemma 2.2; otherwise, $\frac{\nabla_r \text{dist}_p}{|\nabla_r \text{dist}_p|}$ coincides with the farthest direction to $\uparrow_r^p$ by Lemma 2.2, and thus by Corollary

1.2 we have that $\frac{\nabla_r \text{dist}_p}{|\nabla_r \text{dist}_p|} \in \Sigma_r F$. Then by Theorem 5.1, the gradient curve of dist_p starting from r lies in F .

Conversely, if the gradient curve of dist_p from $r \in F \setminus \{p\}$ lies in F , then the farthest direction $\zeta \in \Sigma_r X$ to $\uparrow_r^p$ satisfies $|\zeta \uparrow_r^p| \leq \frac{\pi}{2}$ by Lemma 2.2 if $\nabla_r \text{dist}_p = 0$; otherwise, $\frac{\nabla_r \text{dist}_p}{|\nabla_r \text{dist}_p|}$ belongs to $\Sigma_r F$ and is just the farthest direction to $\uparrow_r^p$ by Lemma 2.2. Hence, F is quasi-convex by Corollary 1.2. $\square$

Moreover, given $\zeta \in \Sigma_p X$ and $\{p_i\}_{i=1}^\infty$ with $p_i \rightarrow p$ and $\uparrow_{p_i}^{p_i} \rightarrow \zeta$ as $i \rightarrow \infty$, we can define the *radial curve* starting from p with direction ζ by the (reparameterized) limit of gradient curves of dist_p starting from p_i (for details refer to [5]). Then Proposition 5.1 implies the following corollary.

Corollary 5.1. *Let F be a quasi-convex subset in an $X \in \text{Alex}(k)$. Then, for any $p \in F$ and any $\zeta \in \Sigma_p F$, the radial curve starting from p with direction ζ lies in F .*

In Proposition 5.1, for ‘extremal’ case, p can be an arbitrary point in X . As another corollary of Proposition 5.1, we can see that the limit of quasi-convex subsets is also quasi-convex (refer to Lemma 4.1.3 in [5] for ‘extremal’ case).

Corollary 5.2. *Let $\{X_n\}_{n=1}^\infty$ be m -dimensional spaces in $\text{Alex}(\kappa)$, and let F_n be quasi-convex in X_n . If $X_n \xrightarrow{GH} X$ with $F_n \rightarrow F \subset X$ as $n \rightarrow \infty$, then F is also quasi-convex in X .*

Proof. By Proposition 5.1, we just need to verify that, for any two distinct points $p, r \in F$, the gradient curve of dist_p starting from r lies in F . Let $p_n, r_n \in F_n$ with $p_n \rightarrow p$ and $r_n \rightarrow r$ as $n \rightarrow \infty$. It is clear that dist_{p_n} converges to dist_p as $n \rightarrow \infty$. A fundamental fact is that the gradient curve of dist_{p_n} starting from r_n converges to the gradient curve of dist_p starting from r (Lemma 2.1.5 in [5]). Due to the quasi-convexity of F_n , the gradient curve of dist_{p_n} starting from r_n lies in F_n , so the gradient curve of dist_p starting from r lies in F . $\square$

Remark 5.1. If X_n converges to X without collapse (in the Gromov-Hausdorff sense) in Corollary 5.2, then similar to ‘extremal’ case (cf. Section 4.1 in [5]) one can show that F_n also converges to F with respect to induced intrinsic metrics from X_n and X .

In the rest of this section, we will provide three applications of Proposition 5.1.

5.2 Intersection of two quasi-convex subsets

Via Proposition 5.1, we can see that the intersection of two quasi-convex subsets is also quasi-convex.

Proposition 5.2. *Let F and G be two quasi-convex subsets in an $X \in \text{Alex}(k)$. Then $F \cap G$ is also quasi-convex in X ; moreover, $\Sigma_p(F \cap G) = \Sigma_p F \cap \Sigma_p G$ for any $p \in F \cap G$.*

Proof. Since F and G are quasi-convex in X , for any $p, r \in F \cap G$ with $p \neq r$, the gradient curve of dist_p starting from r belongs to both F and G (by Proposition 5.1), which implies that $F \cap G$ is also quasi-convex (by Proposition 5.1 again).

Next, for any $p \in F \cap G$, we show that $\Sigma_p(F \cap G) = \Sigma_p F \cap \Sigma_p G$. It is obvious that $\Sigma_p(F \cap G) \subseteq \Sigma_p F \cap \Sigma_p G$. On the other hand, for any $\xi \in \Sigma_p F \cap \Sigma_p G$, the radial curve starting from p with direction ξ lies in $F \cap G$ (by Corollary 5.1), which implies that $\Sigma_p(F \cap G) \supseteq \Sigma_p F \cap \Sigma_p G$. $\square$

Remark 5.2. It is true that the union of two extremal subsets is also extremal [3]. However, in general, the union of two quasi-convex subsets might not be quasi-convex (e.g., the union of two lines in a plane is not quasi-convex).

Remark 5.3. Let F and G be two extremal subsets in an $X \in \text{Alex}(k)$. Without involving the concept of gradient curve, [3] has proven that both $F \cap G$ and $\overline{F \setminus G}$ are also extremal in X by showing $\Sigma_p(F \cap G) = \Sigma_p F \cap \Sigma_p G$ and $\Sigma_p \overline{F \setminus G} = \overline{\Sigma_p F \setminus \Sigma_p G}$ firstly. However, we cannot give a proof for Proposition 5.2 in such a way (because the condition of ‘quasi-convex’ is much weaker than ‘extremal’). Moreover, so far we cannot either show $\Sigma_p \overline{F \setminus G} = \overline{\Sigma_p F \setminus \Sigma_p G}$ or prove that $\overline{F \setminus G}$ is still quasi-convex if F and G are quasi-convex in X with $\dim(X) \geq 3$.

5.3 Quasi-convex subsets in spherical suspensions

Let $Z \triangleq \{z_1, z_2\} * Y$ with $|z_1 z_2| = \pi$ and $Y \in \text{Alex}(1)$ be a spherical suspension (for details about such suspension structure refer to [2]). As examples of quasi-convex subsets, [6] has shown that if a quasi-convex subset F in Z contains at least two points including z_1 , then $F = \{z_1, z_2\} * (F \cap Y)$. In this paper, we provide a short proof and a stronger version of it via Proposition 5.1.

Proposition 5.3. Let $Z = \{z_1, z_2\} * Y$ with $|z_1 z_2| = \pi$ and $Y \in \text{Alex}(1)$, and let F be a quasi-convex subset in Z containing at least two points. Then either $F \subseteq Y$, or there is $\bar{z}_1, \bar{z}_2$ and $\bar{Y} \in \text{Alex}(1)$ with $|\bar{z}_1 \bar{z}_2| = \pi$ such that $Z = \{\bar{z}_1, \bar{z}_2\} * \bar{Y}$ and $F = \{\bar{z}_1, \bar{z}_2\} * (F \cap \bar{Y})$.

Proof. We first consider a special case where z_1 belongs to F . As mentioned above, $F = \{z_1, z_2\} * (F \cap Y)$ in this case. In fact, for any point $r \in F \setminus \{z_1\}$, $[rz_2]$ is the gradient curve of dist_{z_1} starting from r by the spherical suspension structure of Z , and thus has to lie in F by Proposition 5.1. In particular, $z_2 \in F$. Similarly, if $r \neq z_2$, then $[rz_1]$ also belongs to F ; namely, the minimal geodesic $[z_1 z_2]$ passing r belongs to F . This implies that $F = \{z_1, z_2\} * (F \cap Y)$.

We now can assume that $z_1 \notin F$, $z_2 \notin F$ and $F \cap (Z \setminus Y) \neq \emptyset$. Then we can assume that there is $\bar{z}_1 \in F$ and $\bar{z}_2 \in F$ such that $\frac{\pi}{2} > |z_1 \bar{z}_1| = |z_1 \bar{z}_2| \leq |z_2 \bar{z}_2| = |z_2 F|$. Note that $|\uparrow_{\bar{z}_1}^{\bar{z}_2} \uparrow_{\bar{z}_1}^{z_1}| \leq \frac{\pi}{2}$ by the quasi-convexity of F . So, if $|\bar{z}_1 \bar{z}_2| < \pi$, then the spherical suspension structure of Z guarantees that, for any $[\bar{z}_1 \bar{z}_2]$,

$$|\uparrow_{\bar{z}_2}^{\bar{z}_1} \uparrow_{\bar{z}_2}^{z_1}| < \frac{\pi}{2} \quad \text{and} \quad |\uparrow_{\bar{z}_2}^{\bar{z}_1} \uparrow_{\bar{z}_2}^{z_2}| = \pi - |\uparrow_{\bar{z}_2}^{\bar{z}_1} \uparrow_{\bar{z}_2}^{z_1}|.$$

However, the quasi-convexity of F implies that $|\uparrow_{\bar{z}_2}^{\bar{z}_1} \uparrow_{\bar{z}_2}^{z_2}| \leq \frac{\pi}{2}$, a contradiction. Namely, it has to hold that $|\bar{z}_1 \bar{z}_2| = \pi$ (so z_i and $\bar{z}_i$ lie in a (great) circle of perimeter 2π). Hence, there is $\bar{Y} \in \text{Alex}(1)$ such that $Z = \{\bar{z}_1, \bar{z}_2\} * \bar{Y}$; and thus, similar to the special case above, $F = \{\bar{z}_1, \bar{z}_2\} * (F \cap \bar{Y})$. $\square$

5.4 Fixed point set of an isometry

Proposition 5.4. *Let $X \in \text{Alex}(k)$, and let F be the fixed point set of an isometry on X . Then F is quasi-convex in X .*

Recall that the fixed point set of an isometry on a complete Riemannian manifold is totally geodesic, while a quasi-convex subset in a complete Riemannian manifold must be totally geodesic [6].

Proof. We need only to consider the case where F contains at least two points. Let p and r be arbitrary two distinct points in F . By the uniqueness of the gradient curve of dist_p starting from r , it must be fixed by the isometry. I.e., the gradient curve of dist_p starting from r belongs to F , so F is quasi-convex by Proposition 5.1. $\square$

Remark 5.4. Let γ be the isometry fixing F in Proposition 5.4, and let $p \in F$. Note that there is a naturally induced isometry $\bar{\gamma}$ on $\Sigma_p X$, and $\Sigma_p F$ belongs to the fixed point set $\bar{F}$ of $\bar{\gamma}$. On the other hand, by the uniqueness of radial curve starting from a point with a fixed direction, γ must fix the radial curve starting from p with any direction $\xi \in \bar{F}$. Namely, $\Sigma_p F = \bar{F}$.

Remark 5.5. Let $X \in \text{Alex}(k)$, and let Γ be a compact group which acts on X by isometries with nonempty fixed point set F . In [3], it has been shown that F is extremal as a subset of the orbit space X/Γ (where a key tool is ‘strictly convex hull’). Based on this, [6] has proven that F is quasi-convex in X . Note that we can prove it using the same arguments as in the proof of Proposition 5.4. (We would like to point out that, using the technique of strictly convex hull, one can also see that $\Sigma_p F = \bar{F}$ in Remark 5.4.)

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