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# $L^4$-Bound of the Transverse Ricci Curvature under the Sasaki-Ricci Flow

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**In honor of Professor Xiaochun Rong on his seventieth birthday**

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**Abstract.** In this paper, we show that the uniform $L^4$-bound of the transverse Ricci curvature along the Sasaki-Ricci flow on a compact quasi-regular transverse Fano Sasakian $(2n+1)$-manifold $M$. Then we are able to study the structure of the limit space. As consequences, when $M$ is of dimension five and the space of leaves of the characteristic foliation is of type I, any solution of the Sasaki-Ricci flow converges in the Cheeger-Gromov sense to the unique singular orbifold Sasaki-Ricci soliton and is trivial one if $M$ is transverse $K$-stable. Note that when the characteristic foliation is of type II, the same estimates hold along the conic Sasaki-Ricci flow.

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## 1 Introduction

Let $(M,\eta,\xi,\Phi,g)$ be a compact Sasakian manifold of dimension $2n+1$. If the orbits of the Reeb vector field $\xi$ are all closed and hence circles, then integrates to an isometric $U(1)$ action on $(M,g)$. Since it is nowhere zero this action is locally free, that is, the isotropy

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group of every point in $S$ is finite. If the $U(1)$ action is in fact free then the Sasakian structure is said to be regular. Otherwise, it is said to be quasi-regular. If the orbits of $\xi$ are not all closed, the Sasakian structure is said to be irregular [4]. However, by the second structure theorem [57], any Sasakian structure $(\xi,\eta,\Phi)$ on $(M,g)$ is either quasi-regular or there is a sequence of quasi-regular Sasakian structures $(M,\xi_i,\eta_i,\Phi_i,g_i)$ converging in the compact-open $C^\infty$-topology to $(\xi,\eta,\Phi,g)$. It means that there always exists a quasi-regular Sasakian structure $(\xi,\eta,\Phi)$ on $(M,g)$.

A Sasakian $(2n+1)$-manifold is served as the odd-dimensional analogue of Kähler manifolds. For instance, the Kähler cone of a Sasaki-Einstein 5-manifold is a Calabi-Yau 3-fold. It provides interesting examples of the AdS/CFT correspondence. On the other hand, the class of simply connected, closed, oriented, smooth 5-manifolds is classifiable under diffeomorphism due to Smale-Barden [1,58].

In a compact quasi-regular Sasakian manifold, the Reeb vector field induces a $\mathbb{S}^1$-action which generates the finite isotropy groups. It is the regular free action if the isotropy subgroup of every point is trivial. In general, as in [19], the space of leaves has either the codimension at least two fixed point set of every non-trivial isotropy subgroup or the codimension one fixed point set of some non-trivial isotropy subgroup.

It is our goal to address the related issues on the geometrization and classification problems of quasi-regular Sasakian manifolds of dimension five with foliation singularities [16,19].

Along this spirits, in this paper we will focus on the following Sasaki-Ricci flow

$$\frac{\partial}{\partial t}\omega(t)=\omega(t)-\mathrm{Ric}^T_{\omega(t)}, \qquad \omega(0)=\omega_0 \tag{1.1}$$

which is introduced by Smoczyk-Wang-Zhang [61] to study the existence of Sasaki $\eta$-Einstein metrics on Sasakian manifolds. They showed that the flow has the longtime solution and asymptotic converges to a Sasaki $\eta$-Einstein metric when the basic first Chern class is negative ($c_1^B(M)<0$) or null ($c_1^B(M)=0$). It is wild open when a compact Sasakian $(2n+1)$-manifold is transverse Fano ($c_1^B(M)>0$). In the paper of [17], Collins and Jacob proved that the Sasaki-Ricci flow converges exponentially fast to a Sasaki-Einstein metric if one exists, provided the automorphism group of the transverse holomorphic structure is trivial. In general, by comparing the Kähler-Ricci flow on log Fano varieties as in [2], it is hard to deal with because the space of leaves of the characteristic foliation is a polarized, normal projective variety which endowed with the orbifold structure due to (1.3).

In this note, we will assume that $M$ is a compact quasi-regular transverse Fano Sasakian manifold and the space $Z$ of leaves is well-formed (i.e. $M$ has the foliation singularity of type I) which means its orbifold singular locus and algebro-geometric singular locus coincide, equivalently $Z$ has no branch divisors (see Definition 2.1).

Let $(M,\eta,\xi,\Phi,g)$ be a compact quasi-regular Sasakian $(2n+1)$-manifold and $Z=M/F_\xi$ denote the space of leaves of the characteristic foliation which is well-formed, a normal projective variety with codimension at least two orbifold singularities $\Sigma$. Then by the first structure theorem again, $M$ is a principal $\mathbb{S}^1$-orbibundle ($V$-bundle) over $Z$ which