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L^4-Bound of the Transverse Ricci Curvature under the Sasaki-Ricci Flow

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Abstract. In this paper, we show that the uniform L^4-bound of the transverse Ricci curvature along the Sasaki-Ricci flow on a compact quasi-regular transverse Fano Sasakian $(2n+1)$-manifold M. Then we are able to study the structure of the limit space. As consequences, when M is of dimension five and the space of leaves of the characteristic foliation is of type I, any solution of the Sasaki-Ricci flow converges in the Cheeger-Gromov sense to the unique singular orbifold Sasaki-Ricci soliton and is trivial one if M is transverse K-stable. Note that when the characteristic foliation is of type II, the same estimates hold along the conic Sasaki-Ricci flow.

AMS subject classifications: 53E50, 53C25, 53C12, 14E30

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1 Introduction

Let (M,η,ξ,Φ,g) be a compact Sasakian manifold of dimension $2n+1$. If the orbits of the Reeb vector field ξ are all closed and hence circles, then integrates to an isometric $U(1)$ action on (M,g). Since it is nowhere zero this action is locally free, that is, the isotropy

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group of every point in S is finite. If the $U(1)$ action is in fact free then the Sasakian structure is said to be regular. Otherwise, it is said to be quasi-regular. If the orbits of ξ are not all closed, the Sasakian structure is said to be irregular [4]. However, by the second structure theorem [57], any Sasakian structure (ξ,η,Φ) on (M,g) is either quasi-regular or there is a sequence of quasi-regular Sasakian structures $(M,\xi_i,\eta_i,\Phi_i,g_i)$ converging in the compact-open C^∞-topology to (ξ,η,Φ,g). It means that there always exists a quasi-regular Sasakian structure (ξ,η,Φ) on (M,g).

A Sasakian $(2n+1)$-manifold is served as the odd-dimensional analogue of Kähler manifolds. For instance, the Kähler cone of a Sasaki-Einstein 5-manifold is a Calabi-Yau 3-fold. It provides interesting examples of the AdS/CFT correspondence. On the other hand, the class of simply connected, closed, oriented, smooth 5-manifolds is classifiable under diffeomorphism due to Smale-Barden [1,58].

In a compact quasi-regular Sasakian manifold, the Reeb vector field induces a $\mathbb{S}^1$-action which generates the finite isotropy groups. It is the regular free action if the isotropy subgroup of every point is trivial. In general, as in [19], the space of leaves has either the codimension at least two fixed point set of every non-trivial isotropy subgroup or the codimension one fixed point set of some non-trivial isotropy subgroup.

It is our goal to address the related issues on the geometrization and classification problems of quasi-regular Sasakian manifolds of dimension five with foliation singularities [16,19].

Along this spirits, in this paper we will focus on the following Sasaki-Ricci flow

$$\tfrac{\partial}{\partial t}\omega(t)=\omega(t)-\mathrm{Ric}^T_{\omega(t)}, \qquad \omega(0)=\omega_0 \tag{1.1}$$

which is introduced by Smoczyk-Wang-Zhang [61] to study the existence of Sasaki η-Einstein metrics on Sasakian manifolds. They showed that the flow has the longtime solution and asymptotic converges to a Sasaki η-Einstein metric when the basic first Chern class is negative ($c_1^B(M)<0$) or null ($c_1^B(M)=0$). It is wild open when a compact Sasakian $(2n+1)$-manifold is transverse Fano ($c_1^B(M)>0$). In the paper of [17], Collins and Jacob proved that the Sasaki-Ricci flow converges exponentially fast to a Sasaki-Einstein metric if one exists, provided the automorphism group of the transverse holomorphic structure is trivial. In general, by comparing the Kähler-Ricci flow on log Fano varieties as in [2], it is hard to deal with because the space of leaves of the characteristic foliation is a polarized, normal projective variety which endowed with the orbifold structure due to (1.3).

In this note, we will assume that M is a compact quasi-regular transverse Fano Sasakian manifold and the space Z of leaves is well-formed (i.e. M has the foliation singularity of type I) which means its orbifold singular locus and algebro-geometric singular locus coincide, equivalently Z has no branch divisors (see Definition 2.1).

Let (M,η,ξ,Φ,g) be a compact quasi-regular Sasakian $(2n+1)$-manifold and $Z=M/F_\xi$ denote the space of leaves of the characteristic foliation which is well-formed, a normal projective variety with codimension at least two orbifold singularities Σ. Then by the first structure theorem again, M is a principal $\mathbb{S}^1$-orbibundle (V-bundle) over Z which is also a $\mathbb{Q}$-factorial, polarized, normal projective variety such that there is an orbifold Riemannian submersion

$$\pi:(M,g)\to(Z,\omega) \tag{1.2}$$

and

$$K_M^T=\pi^*(K_Z^{orb}). \tag{1.3}$$

If the orbifold structure of the leave space Z is well-formed, then the orbifold canonical divisor K_Z^{orb} and canonical divisor K_Z are the same and thus

$$K_M^T=\pi^*(\varphi^*K_Z)$$

with respect to the orbifold charts on (Z,U_i,φ_i).

In this paper, we will obtain the crucial estimate on the L^4-Bound of the transverse Ricci curvature under the Sasaki-Ricci flow (1.1) in a compact quasi-regular transverse Fano Sasakian manifold $(M,\xi,\eta_0,\Phi_0,g_0,\omega_0)$ of dimension five with foliation singularities of type I. In our upcoming paper [18], we shall deal with the same issue under the so-called conic Sasaki-Ricci flow in a compact quasi-regular transverse Fano Sasakian manifold of dimension five with foliation singularities of type II.

Theorem 1.1. *Let $(M,\xi,\eta_0,\Phi_0,g_0,\omega_0)$ be a compact transverse Fano quasi-regular Sasakian $(2n+1)$-manifold and its the space Z of leaves of the characteristic foliation be well-formed.* **Then,** *under the Sasaki-Ricci flow* (1.1), *there exists a positive constant C such that*

$$\int_M |\mathrm{Ric}^T_{\omega(t)}|^4\omega(t)^n\wedge\eta_0\le C, \tag{1.4}$$

for all $t\in[0,\infty)$. That is it suffices to show that

$$\int_M |\nabla^T\overline{\nabla}^T u(t)|^4\omega(t)^n\wedge\eta_0\le C,$$

for all $t\ge 0$ and for some constant C independent of t. Here $u(t)$ is the evolving transverse Ricci potential.

Furthermore, we will show that the transverse Ricci potentials $u(t)$ which is a basic function behaves very well as $t\to\infty$ under the Sasaki-Ricci flow. This implies that the limit of Sasaki-Ricci flow should be the Sasaki-Ricci soliton in the L^2-topology.

Theorem 1.2. *Let $(M,\xi,\eta_0,\Phi_0,g_0,\omega_0)$ be a compact transverse Fano quasi-regular Sasakian $(2n+1)$-manifold and its the space Z of leaves of the characteristic foliation be well-formed.* **Then,** *under the Sasaki-Ricci flow*

$$\int_M |\nabla^T\nabla^T u|^2\omega(t)^n\wedge\eta_0\to 0,$$
$$\int_M (\Delta_B u-|\nabla^T u|^2+u-a)^2\omega(t)^n\wedge\eta_0\to 0$$

as $t\to\infty$. Here $a(t)=\frac{1}{V}\int_M ue^{-u}\omega(t)^n\wedge\eta_0$.

Now we are able to study the structure of the limit space:

Theorem 1.3. *Let $(M_i,\eta_i,\xi,\Phi_i,g_i)$ be a sequence of quasi-regular Sasakian $(2n+1)$-manifolds with Sasaki metrics $g_i=g_i^T+\eta_i\otimes\eta_i$ such that for basic potentials φ_i*

$$\eta_i=\eta+d_B^c\varphi_i$$

and

$$d\eta_i=d\eta+\sqrt{-1}\partial_B\overline{\partial}_B\varphi_i.$$

We denote that $(Z_i,h_i,J_i,\omega_{h_i})$ are a sequence of well-formed normal projective orbifolds surfaces which are the corresponding foliation leave space with respect to $(M_i,\eta_i,\xi,\Phi_i,g_i)$ such that

$$\omega_{g_i^T}=\frac{1}{2}d\eta_i=\pi^*(\omega_{h_i});\ \Phi_i=\pi^*(J_i)$$

Suppose that $(M_i,\eta_i,\xi,\Phi_i,g_i)$ is a compact smooth transverse Fano Sasakian $(2n+1)$-manifolds satisfying

$$\int_M|\mathrm{Ric}^T_{g_i^T}|^p\omega_i^n\wedge\eta\leq\Lambda,$$

and

$$\mathrm{Vol}(\mathrm{B}_{\xi,g_i^T}(x_i,1))\geq\nu$$

for some $p>n$, $\Lambda>0$, $v>0$. Then, after passing to a subsequence if necessary, (M_i,Φ_i,g_i,x_i) converges in the Cheeger-Gromov sense to limit length spaces $(M_\infty,\Phi_\infty,d_\infty,x_\infty)$ and then $(Z_i,J_i,h_i,\pi(x_i))$ converges to $(Z_\infty,J_\infty,h_\infty,\pi(x_\infty))$ such that for any $r>0$ and $p_i\in M_i$ with $p_i\to p_\infty\in M_\infty$,

$$\begin{aligned}&\mathrm{Vol}(B_{h_i}(\pi(p_i),r))\to\mathcal{H}^{2n}(B_{h_\infty}(\pi(p_\infty),r)),\\&\mathrm{Vol}(B_{\xi,g_i^T}(p_i,r))\to\mathcal{H}^{2n}(B_{\xi,g_\infty^T}(p_\infty,r)).\end{aligned}$$

Moreover,

$$\mathrm{Vol}(B(p_i,r))\to\mathcal{H}^{2n+1}(B(p_\infty,r))$$

where $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure and

1. *M_∞ is a S^1-orbibundle over the normal projective variety $Z_\infty=M_\infty/\mathcal{F}_\xi$.*

2. *Moreover, $Z_\infty=\mathcal{R}\cup\mathcal{S}$ such that $\mathcal{S}$ is a closed singular set of codimension at least two and $\mathcal{R}$ consists of points whose tangent cones are $\mathbb{R}^{2n}$.*

3. *Furthermore, the convergence on the regular part of M_∞ which is a S^1-principle bundle over $\mathcal{R}$ in the $(C^\alpha\cap L_B^{2,p})$-topology for any $0<\alpha<2-\frac{n}{p}$.*

In this note, the central issue is to show the L^4-bound of the transverse Ricci curvature (Theorem 1.1) along the Sasaki-Ricci flow. Then the transverse Ricci potentials $u(t)$ which is a basic function behaves very well as $t\to\infty$ under the Sasaki-Ricci flow. This implies that the structure of the limit of Sasaki-Ricci flow should be the Sasaki-Ricci soliton in the L^2-topology.

With its applications, we will state it without the detailed proof as in the final section. We refer to [12] for some details.

2 Structure theorem and foliation singularities

In this section, we will recall some preliminaries for Sasakian manifolds with foliation singularities, a type II deformation of the Sasakian structure and the Sasaki-Ricci flow. We refer to [4,19,35,59], and references therein for some details.

A Riemannian $(2n+1)$-manifold (M,g,∇) is called a Sasaki manifold if the metric cone $(C(M),\overline{g},J):=(\mathbb{R}^+\times M,\, dr^2+r^2g,J)$ is Kähler with the Kähler form $\overline{\omega}=\frac{1}{2}\sqrt{-1}\partial\overline{\partial}r^2$ and

$$\overline{\eta}=\frac{1}{2}\overline{g}(\xi,\cdot)\qquad\text{and}\qquad\overline{\xi}=J(r\frac{\partial}{\partial r}).$$

The function $\frac{1}{2}r^2$ is hence a global Kähler potential for the cone metric. As $[r=1]=\{1\}\times M\subset C(M)$, we may define the Reeb vector field ξ on M by $\xi=J(\frac{\partial}{\partial r})$ and the contact 1-form η on TM by $\eta=g(\xi,\cdot)$. Then ξ is the unit Killing vector field such that $\eta(\xi)=1$ and $d\eta(\xi,\cdot)=0$. The tensor field of type (1,1), defined by

$$\Phi(Y)=\nabla_Y\xi$$

satisfies the condition $(\nabla_X\Phi)(Y)=g(\xi,Y)X-g(X,Y)\xi$ for any pair of vector fields X and Y on M. Then such a triple (η,ξ,Φ) is called a Sasakian structure on a Sasakian manifold (M,g). The first structure theorem on Sasakian manifolds states that

Proposition 2.1 ([4,42,45,57])**.** *Let (M,η,ξ,Φ,g) be a compact quasi-regular Sasakian manifold of dimension $2n+1$ and Z denote the space of leaves of the characteristic foliation $\mathcal{F}_\xi$ (just as topological space). Then*

1. *Z carries the structure of a Hodge orbifold $\mathcal{Z}=(Z,\Delta)$ with an orbifold Kähler metric h and the Kähler form ω_h which defines an integral class in $H^2_{orb}(Z,\mathbb{Z})$ in such a way that $\pi:(M,g)\to(Z,h,\omega_h)$ is an orbifold Riemannian submersion, and a principal $\mathbb{S}^1$-orbibundle (V-bundle) over Z. Furthermore, it satisfies $\frac{1}{2}d\eta=\pi^*(\omega_h)$. The fibers of π are geodesics.*

2. *Z is also a $\mathbb{Q}$-factorial, polarized, normal projective algebraic variety.*

3. *The orbifold Z is Fano if and only if $\mathrm{Ric}_g>-2$. In this case Z as a topological space is simply connected; and as an algebraic variety is uniruled with Kodaira dimension $-\infty$.*

4. *(M,ξ,g) is Sasaki-Einstein if and only if (Z,h) is Kähler-Einstein with scalar curvature $4n(n+1)$.*

5. *If (M,η,ξ,Φ,g) is regular, then the orbifold structure is trivial and π is a principal circle bundle over a smooth projective algebraic variety.*

For all previous discussions with the special case for $n=2$, we have the following concerning its foliation singularities:

Definition 2.1 ([19]). 1. *Foliation singularities of type I: Let (M,η,ξ,Φ,g) be a compact quasi-regular Sasakian 5-manifold and its leave space $(Z,\emptyset)$ of the characteristic foliation be well-formed. Then Z is a $\mathbb{Q}$-factorial normal projective algebraic orbifold surface with isolated singularities of a finite cyclic quotient of $\mathbb{C}^2$. Accordingly, $p\in Z$ is analytically isomorphic to $p\in Z\simeq(0\in\mathbb{C}^2)/\mu_{Z_r}$,where Z_r is a cyclic group of order r and its action on such open affine neighborhood is defined by*

$$\mu_{Z_r}:(z_1,z_2)\to(\zeta^a z_1,\zeta^b z_2),$$

where ζ is a primitive r-th root of unity. We denote the cyclic quotient singularity by $\frac{1}{r}(a,b)$ with $(a,r)=1=(b,r)$. In particular, the action can be rescaled so that every cyclic quotient singularity corresponds to a $\frac{1}{r}(1,a)$-point with $(r,a)=1$. It is klt (Kawamata log terminal) singularities.

2. *Foliation singularities of type II: Let (M,η,ξ,Φ,g) be a compact quasi-regular Sasakian 5-manifold and its leave space (Z,Δ) has the codimension one fixed point set of some non-trivial isotropy subgroup. In this case, the action*

$$\mu_{Z_r}:(z_1,z_2)\to(e^{\frac{2\pi a_1 i}{r_1}}z_1,e^{\frac{2\pi a_2 i}{r_2}}z_2),$$

for some positive integers r_1, r_2 whose least common multiple is r, and a_i, $i=1,2$ are integers coprime to r_i, $i=1,2$. Then the foliation singular set contains some 3-dimensional Sasakian submanifolds of M. More precisely, the corresponding singularities in (M,η,ξ,Φ,g) is called the Hopf $\mathbb{S}^1$-orbibundle over a Riemann surface Σ_h.

3 The Sasaki-Ricci flow

Let (M,η,ξ,Φ,g) be a compact Sasakian $(2n+1)$-manifold with $g(\xi,\xi)=1$ and the integral curves of ξ be geodesics. For any point $p\in M$, we can construct local coordinates in a neighborhood of p which are simultaneously foliated and Riemann normal coordinates [37]. That is, we can find Riemann normal coordinates $\{x,z^1,\cdots,z^n\}$ on a neighborhood U of p, such that $\frac{\partial}{\partial x}=\xi$ on U. Let $\{U_\alpha\}_{\alpha\in A}$ be an open covering of the Sasakian manifold and $\pi_\alpha:U_\alpha\to V_\alpha\subset\mathbb{C}^n$ be submersions such that $\pi_\alpha\circ\pi_\beta^{-1}:\pi_\beta(U_\alpha\cap U_\beta)\to\pi_\alpha(U_\alpha\cap U_\beta)$ is biholomorphic. On each V_α, there is a canonical isomorphism $d\pi_\alpha:D_p\to T_{\pi_\alpha(p)}V_\alpha$ for any $p\in U_\alpha$, where $D=\ker\xi\subset TM$. Since ξ generates isometries, the restriction of the Sasakian metric g to D gives a well-defined Hermitian metric g_α^T on V_α. This Hermitian metric in fact is Kähler. More precisely, let $z^1,\cdots,z^n$ be the local holomorphic coordinates on V_α. We pull back these to U_α and still write the same for the horizontal distribution D locally. Let x be the coordinate along the leaves with $\xi=\frac{\partial}{\partial x}$. Then we have the foliation local coordinate $\{x,z^1,\cdots,z^n\}$ on U_α and $(D\otimes\mathbb{C})$ is spanned by the fields $Z_j=\left(\frac{\partial}{\partial z^j}+\sqrt{-1}h_j\frac{\partial}{\partial x}\right)$, $j\in\{1,\cdots,n\}$ with $\eta=dx-\sqrt{-1}h_jdz^j+\sqrt{-1}h_{\bar{j}}d\bar{z}^j$ and its dual frame $\{\eta,dz^j,\ j=1,\cdots,n\}$. Here h is a basic function such that $\frac{\partial h}{\partial x}=0$ and $h_j=\frac{\partial h}{\partial z^j}$, $h_{j\bar{l}}=\frac{\partial^2 h}{\partial z^j\partial\bar{z}^l}$ with the foliation normal coordinate $h_j(p)=0$, $h_{j\bar{l}}(p)=\delta_j^l$, $dh_{j\bar{l}}(p)=0$. Moreover, we have $d\eta(Z_\alpha,\overline{Z_\beta})=d\eta(\frac{\partial}{\partial z^\alpha},\frac{\partial}{\partial \bar{z}^\beta})$.Then the Kähler 2-form ω_α^T of the Hermitian metric g_α^T on V_α, which is the same as the restriction of the Levi form $d\eta$ to $\widetilde{D_\alpha^n}$, the slice $\{x=\text{constant}\}$ in U_α, is closed. The collection of Kähler metrics $\{g_\alpha^T\}$ on $\{V_\alpha\}$ is so-called a transverse Kähler metric. We often refer to $d\eta$ as the Kähler form of the transverse Kähler metric g^T in the leaf space $\widetilde{D^n}$.

The Kähler form $d\eta$ on D and the Kähler metric g^T are defined such that $g=g^T+\eta\otimes\eta$. Now in terms of the normal coordinate, we have

$$g^T=g^T_{i\bar{j}}dz^id\bar{z}^j.$$

Here $g^T_{i\bar{j}}=g^T(\frac{\partial}{\partial z^i},\frac{\partial}{\partial \bar{z}^j})$. The transverse Ricci curvature Ric^T of the Levi-Civita connection ∇^T associated to g^T is defined by $\mathrm{Ric}^T=\mathrm{Ric}+2g^T$ and then $R^T=R+2n$. The transverse Ricci form is defined to be $\rho^T=\mathrm{Ric}^T(\Phi\cdot,\cdot)=-\sqrt{-1}R^T_{i\bar{j}}dz^i\wedge d\bar{z}^j$ with

$$R^T_{i\bar{j}}=-\frac{\partial^2}{\partial z^i\partial\bar{z}^j}\log\det(g^T_{\alpha\bar{\beta}})$$

and it is a closed basic (1,1)-form $\rho^T=\rho+2d\eta$.

We recall that a p-form γ on a Sasakian $(2n+1)$-manifold is called basic if $i(\xi)\gamma=0$ and $\mathcal{L}_\xi\gamma=0$. Let Λ_B^p be the sheaf of germs of basic p-forms and Ω_B^p be the set of all global sections of Λ_B^p. It is easy to check that $d\gamma$ is basic if γ is basic. Set $d_B=d|_{\Omega_B^p}$. Then $d_B:=\partial_B+\bar{\partial}_B:\Omega_B^p\to\Omega_B^{p+1}$ with $\partial_B:\Lambda_B^{p,q}\to\Lambda_B^{p+1,q}$ and $\bar{\partial}_B:\Lambda_B^{p,q}\to\Lambda_B^{p,q+1}$. Moreover

$$d_Bd_B^c=\sqrt{-1}\partial_B\bar{\partial}_B \qquad \text{and} \qquad d_B^2=(d_B^c)^2=0$$

for $d_B^c:=\frac{1}{2}\sqrt{-1}(\bar{\partial}_B-\partial_B)$. The basic Laplacian is defined by $\Delta_B:=d_Bd_B^*+d_B^*d_B$. Then we have the basic de Rham complex (Ω_B^*,d_B) and the basic Dolbeault complex $(\Omega_B^{p,*},\bar{\partial}_B)$ and its cohomology ring $H_B^*(\mathcal{F}_\xi)\triangleq H_B^*(M,\mathbb{R})$ of the foliation $\mathcal{F}_\xi$ [33]. Then we can define the orbifold cohomology of the leaf space $Z=M/U(1)$ to be this basic cohomology ring $H^*_{orb}(Z,\mathbb{R})\triangleq H_B^*(F_\xi)$ and the basic first Chern class $c_1^B(M)$ by $c_1^B=[\frac{\rho^T}{2\pi}]_B$. And a transverse Kähler-Einstein metric (or a Sasaki η-Einstein metric) means that it satisfies $[\rho^T]_B=\varkappa[d\eta]_B$ for $\varkappa=-1,0,1$, up to a D-homothetic deformation.

Now we consider the type II deformations of Sasakian structures (M,η,ξ,Φ,g) as followings: By fixing the ξ and varying η, define $\widetilde{\eta}=\eta+d_B^c\varphi$, for $\varphi\in\Omega_B^0$. Then

$$d\widetilde{\eta}=d\eta+\sqrt{-1}\partial_B\overline{\partial}_B\varphi \quad \text{and} \quad \widetilde{\omega}=\omega+\sqrt{-1}\partial_B\overline{\partial}_B\varphi.$$

Hence we have the same transversal holomorphic foliation but with the new Kähler structure on the Kähler cone $C(M)$ and new contact bundle $\widetilde{D}$ with $\widetilde{\omega}=\frac{1}{2}dd^c\widetilde{r}^2$, $\widetilde{r}=re^{\varphi}$. Since $r\frac{\partial}{\partial r}=\widetilde{r}\frac{\partial}{\partial \widetilde{r}}$ and $\xi+\sqrt{-1}r\frac{\partial}{\partial r}=\xi-\sqrt{-1}J(\xi)$ is a holomorphic vector field on $C(M)$, so we have the same holomorphic structure. Finally, by the $\partial_B\overline{\partial}_B$-lemma in the basic Hodge decomposition, there is a basic function $F:M\to\mathbb{R}$ such that

$$\rho^T(x,t)-\varkappa d\eta(x,t)=d_Bd_B^cF=\sqrt{-1}\partial_B\overline{\partial}_BF.$$

Now we focus on finding a new η-Einstein Sasakian structure $(M,\xi,\widetilde{\eta},\widetilde{\Phi},\widetilde{g})$ with $\widetilde{g}^T=(g^T_{i\bar{j}}+\varphi_{i\bar{j}})dz^id\bar{z}^j$ such that

$$\widetilde{\rho}^T=\varkappa d\widetilde{\eta}.$$

Hence $\widetilde{\rho}^T-\rho^T=\varkappa d_Bd_B^c\varphi-d_Bd_B^cF$. It follows that there is a Sasakian analogue of the Monge-Ampère equation for the orbifold version of Calabi-Yau Theorem

$$\frac{\det(g^T_{\alpha\overline{\beta}}+\varphi_{\alpha\overline{\beta}})}{\det(g^T_{\alpha\overline{\beta}})}=e^{-\varkappa\varphi+F}. \tag{3.1}$$

And we consider the Sasaki-Ricci flow on $M\times[0,T)$

$$\frac{d}{dt}g^T(x,t)=-(\mathrm{Ric}^T(x,t)-\varkappa g^T(x,t))$$

which is equivalent to

$$\frac{d}{dt}\varphi=\log\det(g^T_{\alpha\overline{\beta}}+\varphi_{\alpha\overline{\beta}})-\log\det(g^T_{\alpha\overline{\beta}})+\varkappa\varphi-F. \tag{3.2}$$

4 L^4-Bound of the transverse Ricci curvature under the Sasaki-Ricci flow

In this section, we show the L^4-bound of the transverse Ricci curvature under the Sasaki-Ricci flow.

We follow the line in [68] and [16] to prove this estimate. Note that the flow (1.1) can be expressed locally as a parabolic Monge-Ampère equation on a basic Kähler potential φ as in (3.2):

$$\frac{d}{dt}\varphi=\log\det(g^T_{\alpha\overline{\beta}}+\varphi_{\alpha\overline{\beta}})-\log\det(g^T_{\alpha\overline{\beta}})+\varphi-u(0). \tag{4.1}$$

Here $u(0)$ is the transverse Ricci potential of η_0, defined by

$$R^T_{kl} - g^T_{kl} = \partial_k \overline{\partial}_l u(0)$$

which we normalize so that $\frac{1}{V}\int_M e^{-u(0)}\omega_0^n \wedge \eta_0 = 1$. Let $u(t)$ be the evolving transverse Ricci potential. Then

$$\partial_k \overline{\partial}_l \dot{\varphi} = \frac{\partial}{\partial t} g^T_{kl} = g^T_{kl} - R^T_{kl} = \partial_k \overline{\partial}_l u. \tag{4.2}$$

It follows from this equation that φ evolves by $\dot{\varphi}(t) = u(t) + c(t)$, for $c(t)$ depending only on time t. Then by using $c(t)$ to adjust the initial value $\varphi(0)$, we always assume that

$$\varphi(0) = c_0 := \frac{1}{V}\int_0^\infty e^{-t} ||\nabla^T \dot{\varphi}(t)||^2_{L^2} dt + \frac{1}{V}\int_M u(0)\omega_0^n \wedge \eta_0.$$

Since

$$\partial_k \overline{\partial}_l \left(\frac{\partial u}{\partial t}\right) = \partial_k \overline{\partial}_l u + \partial_k \overline{\partial}_l \Delta_B u,$$

then we can

$$a(t) = \frac{1}{V}\int_M u e^{-u} \omega(t)^n \wedge \eta_0$$

such that

$$\frac{\partial u}{\partial t} = \Delta_B u + u - a.$$

Now by Jensen's inequality, we have $a(t) \leq 0$ and then there exist a uniform positive constant C_1 such that [21]

$$-C_1 \leq a(t) \leq 0 \tag{4.3}$$

for all $t \geq 0$. Moreover, it follows from the Poincaré type inequality, one can show that $a(t)$ increases along the Sasaki-Ricci flow, so we may assume

$$\lim_{t\to\infty} a(t) = a_\infty.$$

It follows from [21] that

Lemma 4.1. *Let (M^{2n+1}, ξ, g_0) be a compact Sasakian manifold and let $g^T(t)$ be the solution of the Sasaki-Ricci flow (1.1) with the initial transverse metric g_0^T. Then there exists C depending only on the initial metric such that*

$$||u(t)||_{C^0} + ||\nabla^T u(t)||_{C^0} + ||\Delta_B u(t)||_{C^0} \leq C$$

for all $t \geq 0$.

In order to prove the L^4 bound of transverse Ricci curvature (1.4) under the normalized Sasaki-Ricci flow, it suffices to show that

$$\int_M |\nabla^T\overline{\nabla}^T u(t)|^4\omega(t)^n\wedge\eta_0\leq C, \tag{4.4}$$

for all $t\geq 0$ and for some constant C independent of t. We need the following Lemmas.

Lemma 4.2. *There exists a positive constant $C=C(g_0^T)$ such that*

$$\int_M [|\nabla^T\overline{\nabla}^T u|^2+|\nabla^T\nabla^T u|^2+|\mathrm{Rm}^T|^2]\omega(t)^n\wedge\eta_0\leq C, \tag{4.5}$$

for all $t\in[0,\infty)$.

Proof. Applying the integration by parts, we have

$$\int_M [|\nabla^T\overline{\nabla}^T u|^2\omega(t)^n\wedge\eta_0=\int_M (\Delta_B u)^2\omega(t)^n\wedge\eta_0,$$

and also

$$\begin{aligned}&\int_M |\nabla^T\nabla^T u|^2\omega(t)^n\wedge\eta_0\\ =&\int_M [(\Delta_B u)^2-\left\langle \mathrm{Ric}^T,\partial_B u\overline{\partial}_B u\right\rangle]\omega(t)^n\wedge\eta_0\\ =&\int_M [(\Delta_B u)^2-|\nabla^T u|^2+\left\langle \partial_B\overline{\partial}_B u,\partial_B u\overline{\partial}_B u\right\rangle]\omega(t)^n\wedge\eta_0\\ \leq&\int_M [(\Delta_B u)^2+|\nabla^T\overline{\nabla}^T u|^2+|\nabla^T u|^4]\omega(t)^n\wedge\eta_0.\end{aligned}$$

Moreover, the L^2-bound of the transverse Riemannian curvature tensor follows from uniformly bound of the transverse scalar curvature and the Sasaki analogue of the Chern-Weil theory:

$$\begin{aligned}&\frac{(2\pi)^2}{2^{n-2}(n-2)!}\int_M [2c_2^B-\frac{n}{n+1}(c_1^B)^2]\wedge\omega(t)^{n-2}\wedge\eta_0\\ =&\frac{1}{2^n n!}\int_M [|\mathrm{Rm}^T|-\frac{2}{n(n+1)}(R^T)^2-\frac{(n-1)(n+2)}{n(n+1)}((R^T)^2+(2n(n+1))^2]\omega(t)^n\wedge\eta_0.\end{aligned}$$ □

The following integral inequalities hold by using Lemma 4.1 and integration by parts.

Lemma 4.3. *There exists a universal positive constant $C=C(g_0^T)$ such that*

$$\begin{aligned}&\int_M |\nabla^T\overline{\nabla}^T u|^4\omega^n\wedge\eta_0\leq C\int_M [|\overline{\nabla}^T\nabla^T\nabla^T u|^2+|\nabla^T\nabla^T\overline{\nabla}^T u|^2]\omega(t)^n\wedge\eta_0, &(4.6)\\ &\int_M |\nabla^T\nabla^T u|^4\omega^n\wedge\eta_0\leq C\int_M [|\overline{\nabla}^T\nabla^T\nabla^T u|^2+|\nabla^T\nabla^T\overline{\nabla}^T u|^2\end{aligned}$$

$$+|\nabla^T\nabla^T\nabla^T u|^2]\omega(t)^n\wedge\eta_0, \tag{4.7}$$

and

$$\begin{aligned}&\int_M[|\overline{\nabla}^T\nabla^T\nabla^T u|^2+|\nabla^T\nabla^T\overline{\nabla}^T u|^2+|\nabla^T\nabla^T\nabla^T u|^2]\omega(t)^n\wedge\eta_0\\ \le& C\int_M[|\nabla^T\Delta^T u|^2+|\nabla^T\nabla^T u|^2+|\mathrm{Rm}^T|^2]\omega(t)^n\wedge\eta_0\end{aligned} \tag{4.8}$$

for all $t\in[0,\infty)$.

Now we can prove a uniform bound of $\int_M|\nabla^T\overline{\nabla}^T u(t)|^4\omega(t)^n\wedge\eta_0$ under the Sasaki-Ricci flow.

Proposition 4.1. *There exists a positive constant* $C=C(g_0^T)$ *such that*

$$\begin{aligned}&\int_M[|\overline{\nabla}^T\nabla^T\nabla^T u|^2+|\nabla^T\nabla^T\overline{\nabla}^T u|^2+|\nabla^T\nabla^T\nabla^T u|^2]\omega(t)^n\wedge\eta_0\\ &\quad+\int_M[|\nabla^T\overline{\nabla}^T u|^4+|\nabla^T\nabla^T u|^4]\omega(t)^n\wedge\eta_0\le C,\end{aligned} \tag{4.9}$$

for all $t\in[0,\infty)$.

Proof. From Lemma 4.3, it is sufficient to get a uniform bound of $\int_M|\nabla^T\Delta^T u|^2\omega(t)^n\wedge\eta_0$ under the Sasaki-Ricci flow. Since

$$\left(\frac{\partial}{\partial t}-\Delta_B\right)\Delta_B u=\Delta_B u-|\nabla^T\overline{\nabla}^T u|^2,$$

thus

$$\frac{1}{2}\left(\frac{\partial}{\partial t}-\Delta_B\right)(\Delta_B u)^2=(\Delta_B u)^2-|\nabla^T\Delta_B u|^2-\Delta_B u|\nabla^T\overline{\nabla}^T u|^2.$$

Integrating over the manifold gives

$$\begin{aligned}&\int_M|\nabla^T\Delta_B u|^2\omega^n\wedge\eta_0\\ =&\int_M[(\Delta_B u)^2-\Delta_B u|\nabla^T\overline{\nabla}^T u|^2-\frac{1}{2}\frac{\partial}{\partial t}(\Delta_B u)^2]\omega^n\wedge\eta_0\\ =&\int_M[(\Delta_B u)^2-\Delta_B u|\nabla^T\overline{\nabla}^T u|^2+\frac{1}{2}(\Delta_B u)^3]\omega^n\wedge\eta_0-\frac{1}{2}\frac{d}{dt}\int_M(\Delta_B u)^2\omega^n\wedge\eta_0.\end{aligned}$$

Applying the uniform bound of $\Delta_B u$ from Lemma 4.1 and (4.5), we obtain

$$\int_t^{t+1}\int_M|\nabla^T\Delta_B u|^2\omega(s)^n\wedge\eta_0 ds\le C, \tag{4.10}$$

for all $t\geq 0$. Next we compute

$$
\begin{aligned}
&\frac{d}{dt}\int_M|\nabla^T\Delta_B u|^2\omega^n\wedge\eta_0\\
=&\int_M[|\nabla^T\Delta_B u|^2-|\nabla^T\overline{\nabla}^T\Delta_B u|^2-|\nabla^T\nabla^T\Delta_B u|^2+\Delta_B u|\nabla^T\Delta^T u|^2]\omega^n\wedge\eta_0\\
&\quad-\int_M[\langle\nabla^T|\nabla^T\overline{\nabla}^T u|^2,\nabla^T\Delta_B u\rangle+\langle\overline{\nabla}^T|\nabla^T\overline{\nabla}^T u|^2,\overline{\nabla}^T\Delta_B u\rangle]\omega^n\wedge\eta_0.
\end{aligned}
$$

By integration by parts, we obtain

$$
\begin{aligned}
&\int_M\langle\nabla^T|\nabla^T\overline{\nabla}^T u|^2,\nabla^T\Delta_B u\rangle\omega^n\wedge\eta_0\\
\leq&\frac{1}{4}\int_M[|\nabla^T\overline{\nabla}^T\Delta_B u|^2+|\nabla^T\nabla^T\Delta_B u|^2]\omega^n\wedge\eta_0\\
&\quad+C\int_M[|\nabla^T\Delta_B u|^2+|\nabla^T\nabla^T\overline{\nabla}^T u|^2]\omega^n\wedge\eta_0
\end{aligned}
$$

and also

$$
\begin{aligned}
&\int_M\langle\overline{\nabla}^T|\nabla^T\overline{\nabla}^T u|^2,\overline{\nabla}^T\Delta_B u\rangle\omega^n\wedge\eta_0\\
\leq&\frac{1}{4}\int_M[|\nabla^T\overline{\nabla}^T\Delta_B u|^2+|\nabla^T\nabla^T\Delta_B u|^2]\omega^n\wedge\eta_0\\
&\quad+C\int_M[|\nabla^T\Delta_B u|^2+|\nabla^T\nabla^T\overline{\nabla}^T u|^2]\omega^n\wedge\eta_0.
\end{aligned}
$$

Therefore, by (4.8) and Lemma 4.1,

$$
\begin{aligned}
&\frac{d}{dt}\int_M|\nabla^T\Delta_B u|^2\omega^n\wedge\eta_0\\
\leq&-\frac{1}{2}\int_M[|\nabla^T\overline{\nabla}^T\Delta_B u|^2+|\nabla^T\nabla^T\Delta_B u|^2]\omega^n\wedge\eta_0+C(1+\int_M|\nabla^T\Delta_B u|^2\omega^n\wedge\eta_0)\\
\leq&C(1+\int_M|\nabla^T\Delta_B u|^2\omega^n\wedge\eta_0).
\end{aligned}
$$

The required uniform bound of $\int_M|\nabla^T\Delta_B u|^2\omega^n\wedge\eta_0$ follows from this and (4.10). $\square$

5 Structure of the limit space

In this section, we will show that the transverse Ricci potentials $u(t)$ which is a basic function behaves very well as $t\to\infty$ under the Sasaki-Ricci flow. This implies that the structure of the limit of Sasaki-Ricci flow should be the Sasaki-Ricci soliton in the L^2-topology. We first define

$$\mu(g^T)=\inf\{\mathcal{W}^T(g^T,f):\textstyle\int_M e^{-f}\omega^n\wedge\eta_0=V\},$$

where

$$\mathcal{W}^T(g^T,f)=(2\pi)^{-n}\int_M (R^T+|\nabla^T f|^2+f)e^{-f}\omega^n\wedge\eta_0.$$

Note that,

$$\mu(g^T)\le\frac{1}{V}\int_M ue^{-u}\omega^n\wedge\eta_0.$$

Now under the Sasaki-Ricci flow, for any backward heat equation [21]

$$\frac{\partial f}{\partial t}=-\Delta_B f+|\nabla^T f|^2+\Delta_B u, \tag{5.1}$$

we have

$$\frac{d}{dt}\mathcal{W}^T(g^T,f)=\frac{1}{V}\int_M(|\nabla^T\overline{\nabla}^T(u-f)|^2+|\nabla^T\nabla^T f|^2)e^{-f}\omega^n\wedge\eta_0\ge 0$$

and then

$$\mu(g_0^T)\le\mu(g^T(t))\le 0$$

for all $t\ge 0$.

Lemma 5.1. *Under the Sasaki-Ricci flow, for a smooth basic function f*

$$\int_M|\nabla^T\nabla^T f|^2\omega^n\wedge\eta_0\le C(g_0^T)\int_M|\nabla^T\overline{\nabla}^T f|^2\omega^n\wedge\eta_0.$$

Proof. We may assume that $\int_M fe^{-u}\omega^n\wedge\eta_0=0$. It follows from [21, Theorem 8.1], we have the transverse weighted Poincaré inequality

$$\frac{1}{V}\int_M f^2e^{-u}\omega^n\wedge\eta_0\le\frac{1}{V}\int_M|\nabla^T f|^2e^{-u}\omega^n\wedge\eta_0+(\frac{1}{V}\int_M fe^{-u}\omega^n\wedge\eta_0)^2$$

for all basic function $f\in C_B^\infty(M;\mathbb{R})$. Thus

$$\int_M f^2e^{-u}\omega^n\wedge\eta_0\le\int_M|\nabla^T f|^2e^{-u}\omega^n\wedge\eta_0.$$

It follows from Lemma 4.1 that

$$\int_M f^2\omega^n\wedge\eta_0\le C(g_0)\int_M|\nabla^T f|^2\omega^n\wedge\eta_0.$$

Thus

$$\begin{aligned}\int_M|\nabla^T f|^2\omega^n\wedge\eta_0&=-\int_M f\Delta_B f\omega^n\wedge\eta_0\\&\le\frac{1}{2C}\int_M f^2\omega^n\wedge\eta_0+2C\int_M(\Delta_B f)^2\omega^n\wedge\eta_0\end{aligned}$$

$$\leq \frac{1}{2}\int_M |\nabla^T f|^2 \omega^n \wedge \eta_0 + 2C\int_M (\Delta_B f)^2 \omega^n \wedge \eta_0$$

and

$$\int_M |\nabla^T f|^2 \omega^n \wedge \eta_0 \leq C(g_0)\int_M (\Delta_B f)^2 \omega^n \wedge \eta_0.$$

On the other hand,

$$\begin{aligned}
\int_M |\nabla^T \nabla^T f|^2 \omega^n \wedge \eta_0 &= \int_M ((\Delta_B f)^2 - \mathrm{Ric}^T(\nabla^T f, \nabla^T f))\omega^n \wedge \eta_0 \\
&= \int_M ((\Delta_B f)^2 - |\nabla^T f|^2)\omega^n \wedge \eta_0 + \int_M \nabla_i^T \overline{\nabla}_j^T u \overline{\nabla}_i^T f \nabla_j^T f \omega^n \wedge \eta_0 \\
&= \int_M ((\Delta_B f)^2 - |\nabla^T f|^2)\omega^n \wedge \eta_0 \\
&\quad - \int_M \overline{\nabla}_j^T u(\Delta_B f \nabla_j^T f + \overline{\nabla}_i^T f \nabla_i^T \nabla_j^T f)\omega^n \wedge \eta_0 \\
&\leq \int_M ((\Delta_B f)^2 - |\nabla^T f|^2)\omega^n \wedge \eta_0 \\
&\quad + \int_M ((\Delta_B f)^2 + \frac{1}{2}|\nabla^T \nabla^T f|^2 + |\overline{\nabla}^T u|^2 |\nabla^T f|^2)\omega^n \wedge \eta_0 \\
&\leq \int_M (2(\Delta_B f)^2 + \frac{1}{2}|\nabla^T \nabla^T f|^2 + C|\nabla^T f|^2)\omega^n \wedge \eta_0
\end{aligned}$$

and then

$$\begin{aligned}
\int_M |\nabla^T \nabla^T f|^2 \omega^n \wedge \eta_0 &\leq \int_M (4(\Delta_B f)^2 + 2C|\nabla^T f|^2)\omega^n \wedge \eta_0 \\
&\leq C\int_M (\Delta_B f)^2 \omega^n \wedge \eta_0 \\
&\leq C(g_0)\int_M |\nabla^T \overline{\nabla}^T f|^2 \omega^n \wedge \eta_0.
\end{aligned}$$

□

Theorem 5.1. *Under the Sasaki-Ricci flow*

$$\int_0^\infty \int_M |\nabla^T \nabla^T u|^2 \omega(t)^n \wedge \eta_0 \wedge dt < \infty. \tag{5.2}$$

In particular,

$$\int_M |\nabla^T \nabla^T u|^2 \omega(t)^n \wedge \eta_0 \to 0 \tag{5.3}$$

as $t \to \infty$.

Proof. Let f_k be a minimizer of $\mu(g^T(k))$ with $\int_M e^{-f_k}\omega^n \wedge \eta_0 = V$ and $f_k(t)$ be the solution of the backward heat equation (5.1) on the time interval $[k-1,k]$. Then

$$\frac{1}{V}\int_{k-1}^k \int_M (|\nabla^T \nabla^T f_k|^2 + |\nabla^T \overline{\nabla}^T (u - f_k)|^2) e^{-f_k}\omega^n \wedge \eta_0 \wedge dt \leq \mu(g^T(k)) - \mu(g^T(k-1)).$$

Hence by using $\mu(g^T(t))\leq 0$

$$\sum_{k=1}^{\infty}\int_{k-1}^{k}\int_M (|\nabla^T\nabla^T f_k|^2+|\nabla^T\overline{\nabla}^T(u-f_k)|^2)\omega^n\wedge\eta_0\wedge dt\leq C(g_0^T).$$

On the other hand, by applying the above estimate and Lemma 5.1 to $(u-f_k)$, we have

$$\begin{aligned}
&\int_0^{\infty}\int_M |\nabla^T\nabla^T u|^2\omega(t)^n\wedge\eta_0\wedge dt\\
\leq&\sum_{k=1}^{\infty}\int_M (2|\nabla^T\nabla^T f_k|^2+2|\nabla^T\nabla^T(u-f_k)|^2)\omega^n\wedge\eta_0\wedge dt\\
\leq&\sum_{k=1}^{\infty}\int_M (2|\nabla^T\nabla^T f_k|^2+2C|\nabla^T\overline{\nabla}^T(u-f_k)|^2)\omega^n\wedge\eta_0\wedge dt\\
\leq& C(g_0^T).
\end{aligned}$$

This is the estimate (5.2). Next, it follows from the straight computation that

$$\begin{aligned}
&\frac{\partial}{\partial t}|\nabla^T\nabla^T u|^2\\
=&\Delta_B|\nabla^T\nabla^T u|^2-|\nabla^T\nabla^T\nabla^T u|^2-|\overline{\nabla}^T\nabla^T\nabla^T u|^2-2\mathrm{Rm}^T(\nabla^T\nabla^T u,\overline{\nabla}^T\overline{\nabla}^T u)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}\int_M |\nabla^T\nabla^T u|^2\omega(t)^n\wedge\eta_0\\
\leq&\int_M [(||\nabla u(t)||_{C^0}^2+||\Delta_B u(t)||_{C^0})|\nabla^T\nabla^T u|^2+|\nabla^T\nabla^T\overline{\nabla}^T u|^2\\
&\qquad+||\nabla u(t)||_{C^0}^2|\mathrm{Rm}^T|^2]\omega(t)^n\wedge\eta_0.
\end{aligned}$$

Then, from Lemmas 4.1-4.2 and Proposition 4.1, we have

$$\frac{d}{dt}\int_M |\nabla^T\nabla^T u|^2\omega(t)^n\wedge\eta_0\leq C(g_0^T).$$

Hence

$$\int_M |\nabla^T\nabla^T u|^2\omega(t)^n\wedge\eta_0\to 0$$

as $t\to\infty$. □

Similarly, we have

Theorem 5.2. *Under the Sasaki-Ricci flow,*

$$\int_t^{t+1}\int_M |\nabla^T(\Delta_B u - |\nabla^T u|^2 + u)|^2 \omega(t)^n \wedge \eta_0 \wedge ds \to 0 \tag{5.4}$$

as $t\to\infty$ and then

$$\int_M (\Delta_B u - |\nabla^T u|^2 + u - a)^2 \omega(t)^n \wedge \eta_0 \to 0 \tag{5.5}$$

as $t\to\infty$.

Proof. Note that by the transverse Ricci potential relation

$$\nabla_i^T(\Delta_B u - |\nabla^T u|^2 + u) = \overline{\nabla}_j^T \nabla_i^T \nabla_j^T u - \nabla_i^T \nabla_j^T u \overline{\nabla}_j^T u$$

and then

$$|\nabla^T(\Delta_B u - |\nabla^T u|^2 + u)|^2 \leq 2(|\nabla_i^T \nabla_j^T u|^2 |\overline{\nabla}_j^T u|^2 + |\overline{\nabla}_j^T \nabla_i^T \nabla_j^T u|^2).$$

In order to derive (5.4), it suffices to prove that

$$\int_t^{t+1}\int_M |\overline{\nabla}_j^T \nabla_i^T \nabla_j^T u|^2 \omega(t)^n \wedge \eta_0 \wedge ds \to 0. \tag{5.6}$$

Since the Reeb vector field and the transverse holomorphic structure are both invariant, all the integrands are only involved with the transverse Kähler structure $\omega(t)$ and basic functions $u(t)$. Hence, under the Sasaki-Ricci flow, when one applies integration by parts, the expressions involved behave essentially the same as in the Kähler-Ricci flow. Hence (5.6) follows easily from subsection 4.1 and [68, Proposition 3.2].

Next we denote $\Delta_B u - |\nabla^T u|^2 + u - a = H$ with $\int_M He^{-u}\omega(t)^n \wedge \eta_0 = 0$. Then, by weighted Poincaré inequality ([22, Theorem 1.1]) along the Sasaki-Ricci flow

$$\left(\int_M f^{\frac{2(2n+1)}{2n-1}} \omega(t)^n \wedge \eta_0\right)^{\frac{2n-1}{2n+1}} \leq C_S(g_0^T, n)\int_M (||\nabla^T f||^2 + f^2)\omega(t)^n \wedge \eta_0 \tag{5.7}$$

for every $f \in W_B^{1,2}(M)$ and the uniform bound of u, we have

$$\int_M H^2 \omega(t)^n \wedge \eta_0 \leq C \int_M |\nabla^T H|^2 \omega(t)^n \wedge \eta_0$$

and then

$$\int_t^{t+1}\int_M H^2 \omega(t)^n \wedge \eta_0 \wedge dt \to 0$$

as $t\to\infty$. Since

$$\frac{\partial H}{\partial t} = \Delta_B H + H - \frac{da}{dt} + |\nabla^T \nabla^T u|^2,$$

it follows from Proposition 4.1 that

$$\begin{aligned}\frac{d}{dt}\int_M H^2\omega(t)^n\wedge\eta_0=&\int_M 2H(\Delta_B H+H-\frac{da}{dt}+|\nabla^T\nabla^T u|^2+\frac{1}{2}H\Delta_B u)\omega(t)^n\wedge\eta_0\\ \leq&\int_M 2H(H+|\nabla^T\nabla^T u|^2+\frac{1}{2}H\Delta_B u)\omega(t)^n\wedge\eta_0\\ \leq&(C+|\Delta_B u|^2)\int_M H^2\omega(t)^n\wedge\eta_0+\int_M|\nabla^T\nabla^T u|^4\omega(t)^n\wedge\eta_0\\ \leq&C(1+\int_M H^2\omega(t)^n\wedge\eta_0).\end{aligned}$$

Thus

$$\frac{d}{dt}\int_M H^2\omega(t)^n\wedge\eta_0\leq C(1+\int_M H^2\omega(t)^n\wedge\eta_0)$$

and

$$\int_M(\Delta_B u-|\nabla^T u|^2+u-a)^2\omega(t)^n\wedge\eta_0\to 0$$

as $t\to\infty$. □

Now by the first structure theorem on Sasakian manifolds, M is a principal S^1-orbibundle (V-bundle) over Z which is also a Q-factorial, polarized, normal projective orbifold such that there is an orbifold Riemannian submersion $\pi:(M,g,\omega)\to(Z,h,\omega_h)$ with $g=g^T+\eta\otimes\eta$, $g^T=\pi^*(h)$; $\frac{1}{2}d\eta=\pi^*(\omega_h)$. The orbit ξ_x is compact for any $x\in M$, we then define the transverse distance function as

$$d^T(x,y)\triangleq d_g(\xi_x,\xi_y),$$

where d is the distance function defined by the Sasaki metric g. Then

$$d^T(x,y)=d_h(\pi(x),\pi(y)).$$

We define a transverse ball $B_{\xi,g}(x,r)$ as follows:

$$B_{\xi,g}(x,r)=\left\{y:d^T(x,y)<r\right\}=\{y:d_h(\pi(x),\pi(y))<r\}.$$

Based on Perelman's non-collapsing theorem for a transverse ball along the unnormalizing Sasaki-Ricci flow, it follows that

Lemma 5.2 ([21, Proposition 7.2], [41, Lemma 6.2]). *Let (M^{2n+1},ξ,g_0) be a compact Sasakian manifold and let $g^T(t)$ be the solution of the unnormalizing Sasaki-Ricci flow with the initial transverse metric g_0^T. Then there exists a positive constant C such that for every $x\in M$, if $|S^T|\leq r^{-2}$ on $B_{\xi,g(t)}(x,r)$ for $r\in(0,r_0]$, where r_0 is a fixed sufficiently small positive number, then*

$$\mathrm{Vol}(B_{\xi,g(t)}(x,r))\geq Cr^{2n}.$$

Moreover, the transverse scalar curvature R^T and transverse diameters $d^T_{g^T(t)}$ are uniformly bounded under the Sasaki-Ricci flow. As a consequence, there is a uniform constant C such that

$$\mathrm{diam}(M,g(t))\leq C.$$

Then, Theorem 1.3 follows easily from the Cheeger-Colding-Tian structure theory for Kähler orbifolds [11,68] for a transverse ball along the Sasaki-Ricci flow.

6 Applications

By the convergence theorem in Theorem 1.3, we have the regularity of the limit space: We define a family of Sasaki-Ricci flow $g_i^T(t)$ by

$$(M,g_i^T(t))=(M,g^T(t_i+t))$$

for $t\geq -1$ and $t_i\to\infty$ and for $g_i^T(t)=\pi^*(h_i(t))$

$$(Z,h_i(t))=(M,h_i(t_i+t)).$$

Now for the associated transverse Ricci potential $u_i(t)$ as in Lemma 4.1, we have

$$||u_i(t)||_{C^0}+||\nabla^T u_i(t)||_{C^0}+||\Delta_B u_i(t)||_{C^0}\leq C.$$

Moreover, it follows (5.3) that

$$\int_M |\nabla^T\nabla^T u|^2\omega(t)^n\wedge\eta_0\to 0$$

as $i\to\infty$. Furthermore, by Theorem 1.1 and a convergence Theorem 1.3, passing to a subsequence if necessary, we have at $t=0$,

$$(M,g_i^T(0))\to(M_\infty,g_\infty^T,d_\infty^T)$$

such that

$$(Z,h_i(0))\to(Z_\infty,h_\infty,d_{h_\infty})$$

in the Cheeger-Gromov sense. Moreover,

$$(g_i^T(0),u_i(0))\overset{C^\alpha\cap L_B^{2,p}}{\to}(g_\infty^T,u_\infty) \tag{6.1}$$

on $(M_\infty)_{reg}$ which is a S^1-principle bundle over $\mathcal{R}$. For the solution $(M,\omega(t),g^T(t))$ of the Sasaki-Ricci flow and the line bundle $(K_M^T)^{-1},h(t)=\omega^n(t))$ with the basic Hermitian metric $h(t)$, we work on the evolution of the basic transverse holomorphic line bundle $((K_M^T)^{-m},h^m(t))$ for a large integer m such that $(K_M^T)^{-m}$ is very ample. We consider the basic embedding which is S^1-equivariant with respect to the weighted $\mathbf{C}^*$action in $\mathbf{C}^{N_m+1}$

$$\Psi: M\to(\mathbf{CP}^{N_m},\omega_{FS})$$

defined by the orthonormal basic transverse holomorphic section $\{\sigma_0,\sigma_1,...\sigma_N\}$ in $H_B^0(M,(K_M^T)^{-m})$ with $N_m=\dim H_B^0(M,(K_M^T)^{-m})-1$ with $\int_M(\sigma_i,\sigma_j)_{h^m(t)}\omega^n(t)\wedge\eta_0=\delta_{ij}$.

Define

$$\mathcal{F}_m(x,t):=\sum_{\alpha=0}^{N_m}||\sigma_\alpha||^2_{h^m}(x). \tag{6.2}$$

Note that, under these notations, the curvature form of the Chern connection is $Ric(h(t))=m\omega(t)$. Then the following result is a Sasaki analogue of the partial C^0-estimate which was obtained in the Kähler-Ricci flow [68, Theorem 5.1] and the proof there does carry over to our Sasaki setting due to the first structure theorem again on quasi-regular Sasakian manifolds.

Lemma 6.1. *Suppose* (6.1) *holds, we have*

$$\inf_{t_i}\inf_{x\in M}\mathcal{F}_m(x,t_i)>0 \tag{6.3}$$

for a sequence of $m\to\infty$.

Finally, as a consequence of the first structure theorem for Sasakian manifolds and the partial C^0-estimate [12], the Gromov-Hausdorff limit Z_∞ is a variety embedded in some $\mathbb{CP}^N$ and the singular set $\mathcal{S}$ is a normal subvariety [63, Theorem 1.6]. This will imply the following:

Corollary 6.1 ([12]). *Let (M,ξ,η_0,g_0) be a compact quasi-regular transverse Fano Sasakian manifold of dimension five and $(Z_0=M/\mathcal{F}_\xi,h_0,\omega_{h_0})$ denote the space of leaves of the characteristic foliation which is a normal Fano projective Kähler orbifold surface with codimension two orbifold singularities Σ_0. Then, under the Sasaki-Ricci flow* (1.1)*, $(M(t),\xi,\eta(t),g(t))$ converges to a compact quasi-regular transverse Fano Sasakian orbifold manifold $(M_\infty,\xi,\eta_\infty,g_\infty)$ with the leave space of orbifold Kähler manifold $(Z_\infty=M_\infty/\mathcal{F}_\xi,h_\infty)$ which can have at worst codimension two orbifold singularities Σ_∞. Furthermore, $g^T(t_i)$ converges to a gradient Sasaki-Ricci soliton orbifold metric g^{T}_∞ on M_∞ with $g^{\mathrm{T}}_\infty=\pi^*(h_\infty)$ such that h_∞ is the smooth Kähler-Ricci soliton metric in the Cheeger-Gromov topology on $Z_\infty\backslash\Sigma_\infty$. Furthermore, M_∞ is a $\mathbb{S}^1$-orbibundle over Z_∞ which is a normal projective variety and the singular set $\mathcal{S}$ of Z_∞ is a codimension two orbifold singularities.*

As the final consequence, we will show that the gradient Sasaki-Ricci soliton orbifold metric is a Sasaki-Einstein metric if M is transverse K-stable. This is an old dimensional counterpart of Yau-Tian-Donaldson conjecture on a compact K-stable Kähler manifold [13–15, 66]. It can be viewed as a Sasaki analogue of Tian-Zhang's [68] and Chen-Sun-Wang's result [24] for the Kähler-Ricci flow.

For the Hamiltonian holomorphic vector field V, $d\pi_\alpha(V)$ is a holomorphic vector field on V_α and the complex valued Hamiltonian function $u_V:=\sqrt{-1}\eta(V)$ satisfies $\overline{\partial}_B u_V=-\frac{\sqrt{-1}}{2}i_V d\eta$. Assume we normalize that $c_1^B(M)=[\frac{1}{2}d\eta]_B$, there exists a basic function h_ω such that $Ric^T(x,t)-\omega(x,t)=\sqrt{-1}\partial_B\overline{\partial}_B h_\omega$. The Sasaki-Futaki invariant [5,35]

$$f_M(V)=\int_M V(h_\omega)\omega^n\wedge\eta \tag{6.4}$$

is only depends on the basic cohomology represented by $d\eta$, and not on the particular transverse Kähler metric. It is clear that f_M vanishes if M has a Sasaki-Einstein metric in its basic cohomology class. One also have the following reformulation:

$$f_M(V) = -n\int_M u_V(\mathrm{Ric}^T_\omega - \omega)\omega^{n-1}\wedge\eta = -\int_M u_V(R^T_\omega - n)\omega^n\wedge\eta. \tag{6.5}$$

Let (M,ξ,η,g,ω) be a compact transverse Fano quasi-regular Sasakian manifold and its leave space Z be the normal Fano projective Kähler orbifold and well-formed. Following the notions as in [35] and [32], the Sasaki-Futaki invariant can be extended to the generalized Sasaki-Futaki invariant which has the following reformulation involving (Z,h,ω_h):

$$f_M(V) = f_Z(X) \tag{6.6}$$

with

$$f_Z(X) = -n\int_Z \theta_X(\mathrm{Ric}_{\omega_h} - \omega_h){\omega_h}^{n-1} = -\int_Z \theta_X(R_{\omega_h} - n){\omega_h}^n.$$

By applying first structure theory for a quasi-regular Sasakian manifold, there exists a Riemannian submersion, $\mathbb{S}^1$-orbibundle $\pi: M \to Z$, such that $K^T_M = \pi^*(K^{orb}_Z) = \pi^*(\varphi^* K_Z)$. Then by the CR Kodaira embedding theorem [56], there exists an embedding $\Psi: M \to (\mathbb{CP}^N, \omega_{FS})$ defined by the basic transverse holomorphic section $\{s_0, s_1, \cdots, s_N\}$ of $H^0_B(M, (K^T_M)^{-m})$ which is $\mathbb{S}^1$-equivariant with respect to the weighted $\mathbb{C}^*$-action in $\mathbb{C}^{N+1}$ with $N = \dim H^0_B(M,(K^T_M)^{-m}) - 1$ for a large positive integer m. In fact, in our situation Z is also normal Fano, there is an embedding $\psi_{|mK_Z^{-1}|}: Z \to \mathcal{P}(H^0(Z, K_Z{}^{-m}))$. Define $\Psi_{|m(K^T_M)^{-1}|} = \psi_{|mK_Z^{-1}|} \circ \pi$ such that

$$\Psi_{|m(K^T_M)^{-1}|}: M \to \mathcal{P}(H^0_B(M,(K^T_M)^{-m}).$$

We define $Diff^T(M) = \{\sigma \in Diff(M) \mid \sigma_*\xi = \xi \text{ and } \sigma^* g^T = (\sigma^* g)^T\}$ and $SL^T(N+1,\mathbb{C}) = SL(N+1,\mathbb{C}) \cap Diff^T(M)$.Any other basis of $H^0_B(M,(K^T_M)^{-m})$ gives an embedding of the form $\sigma^T \circ \Psi_{|m(K^T_M)^{-1}|}$ with $\sigma^T \in SL^T(N+1,\mathbb{C})$. Now for any subgroup of the weighted $\mathbb{C}^*$action $G_0 = \{\sigma^T(t)\}_{t\in\mathbb{C}^*}$ of $SL^T(N+1,\mathbb{C})$, there is a unique limiting

$$M_\infty = \lim_{t\to 0} \sigma^T(t)(M) \subset \mathbb{CP}^N.$$

As our application of Corollary 6.1, M_∞ has its leave space $Z_\infty = M_\infty/\mathcal{F}_\xi$ which is a normal projective Kähler orbifold with at worst codimension two orbifold singularities Σ_∞.

Let V be a the Hamiltonian holomorphic vector field whose real part generates the action by $\sigma^T(e^{-s})$. If Z_∞ is normal Fano, there is a generalized Futaki invariant defined by $f_{Z_\infty}(X)$ and then a generalized Sasaki-Futaki invariant defined by $f_{M_\infty}(V)$ as in (6.4), (6.6). Thus one can introduce the Sasaki analogue of the K-stable on Kähler manifolds [28,64,66].

Definition 6.1. *Let (M,ξ,η,g,ω) be a compact transverse Fano quasi-regular Sasakian manifold and its leave space (Z,h,ω_h) be a normal Fano projective Kähler orbifold and well-formed. We say that M is transverse K-stable with respect to $(K_M^T)^{-m}$ if the generalized Sasaki-Futaki invariant*

$$\mathrm{Re} f_{M_\infty}(V) \geq 0 \quad \text{or} \quad \mathrm{Re} f_{Z_\infty}(X) \geq 0$$

for any weighted $\mathbb{C}^*$-action $G_0 = \{\sigma^T(t)\}_{t\in\mathbf{C}^*}$ of $SL^T(N+1,\mathbb{C})$ with a normal Fano $Z_\infty = M_\infty/\mathcal{F}_\xi$ and the equality holds if and only if M_∞ is transverse biholomorphic to M. We say that M is transverse K-stable if it is transverse K-stable for all large positive integer m.

Corollary 6.2 ([12])**.** *Let (M,ξ,η_0,g_0) be a compact quasi-regular transverse Fano Sasakian manifold of dimension five and $(Z_0 = M/\mathcal{F}_\xi, h_0, \omega_{h_0})$ be the space of leaves of the characteristic foliation which is well-formed with codimension two orbifold singularities Σ_0. If M is transverse stable, then under the Sasaki-Ricci flow, $M(t)$ converges to a compact transverse Fano Sasakian manifold M_∞ which is isomorphic to M endowed with a smooth Sasaki-Einstein metric.*

Remark 6.1.

1. Note that by continuity method, Collins and Székelyhidi [27] showed that a polarized affine variety admits a Ricci-flat Kähler cone metric if and only if it is K-stable. In particular, the Sasakian manifold admits a Sasaki-Einstein metric if and only if its Kähler cone is K-stable.
2. On the other hand, instead of K-stability on its Kähler cone, one can have the so-called transverse K-stability on a compact quasi-regular transverse Fano Sasakian manifold with the space of leaves of the characteristic foliation which is well-formed and also a normal Fano projective Kähler orbifold.
3. In the upcoming paper [18], under the conic Sasaki-Ricci flow, we will prove the conic version of Yau-Tian-Donaldson conjecture on a log transverse Fano Sasakian manifold in which its leave space Z_0 is not well-formed. It means that the orbifold structure (Z_0,Δ) has the codimension one fixed point set of some non-trivial isotropy subgroup.

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