J. Math. Study
doi: 10.4208/jms.v58n1.25.04

Vol. **58**, No. 1, pp. 62-81
March 2025

Affine Isoperimetric Type Inequalities for Static Convex Domains in Hyperbolic Space

Yingxiang Hu[1], Haizhong Li[2], Yao Wan[3,*] and Botong Xu[4]

[1] *School of Mathematical Sciences, Beihang University, Beijing 100191, China;*
[2] *Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China;*
[3] *Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong 999077, China;*
[4] *Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel.*

Received September 5, 2024; Accepted December 18, 2024;
Published online March 30, 2025.

In honor of Professor Xiaochun Rong on his seventieth birthday

Abstract. In this paper, the notion of hyperbolic ellipsoids in hyperbolic space is introduced. Using a natural orthogonal projection from hyperbolic space to Euclidean space, we establish affine isoperimetric type inequalities for static convex domains in hyperbolic space. Moreover, equality of such inequalities is characterized by these hyperbolic ellipsoids.

AMS subject classifications: 52A40, 53C24, 53A15

Key words: Static convex, Blaschke-Santaló inequality, affine isoperimetric inequality, orthogonal projection, hyperbolic space.

1 Introduction

The classical Minkowski's second inequality [34, Thm. 7.2.1] states that if $K\subset\mathbb{R}^n$ is a convex body (that is, a compact, convex set with non-empty interior) with smooth boundary ∂K and H is the mean curvature of ∂K, then

$$\mathrm{Area}(\partial K)^2 \geq \frac{n}{n-1}\mathrm{Vol}(K)\int_{\partial K} H dA, \tag{1.1}$$

with equality if and only if K is a ball. This inequality was later generalized by Reilly [31] to compact Riemannian manifolds with nonnegative Ricci curvature and convex boundary. Using the generalized Reilly's formula [30], Xia [40] proved the following Minkowski type inequalities in hyperbolic space.

*Corresponding author. *Email addresses:* huyingxiang@buaa.edu.cn (Hu Y), lihz@tsinghua.edu.cn (Li H), yaowan@cuhk.edu.hk (Wan Y), botongxu@campus.technion.ac.il (Xu B)

Theorem 1.1 ([40]). *Let K be a smooth bounded domain in $\mathbb{H}^n$. Let $V(x)=\cosh r$, where $r(x)=d(x,p_0)$ is the geodesic distance to a fixed point $p_0\in\mathbb{H}^n$. Assume that the second fundamental form of ∂K satisfies*

$$h_{ij}\geq\frac{V_{,\nu}}{V}g_{ij}. \tag{1.2}$$

Then there holds

$$\left(\int_{\partial K}VdA\right)^2\geq\frac{n}{n-1}\int_K V d\mathrm{vol}\cdot\int_{\partial K}VHdA, \tag{1.3}$$

where $d\mathrm{vol}$ is the volume element of $\mathbb{H}^n$. Equality holds if and only if K is a geodesic ball.

The condition (1.2) is called *static convexity*, which was introduced by Brendle and Wang [9] for its correspondence in static space-time. In particular, when K is a smooth bounded domain in $\mathbb{R}^n$, then the weight function $V\equiv1$ and the static convexity (1.2) turns out to be the usual convexity (i.e. $h_{ij}\geq0$), while the inequality (1.3) reduces to the classical Minkowski's second inequality (1.1). Moreover, it was proved in [40] that the equality in (1.3) is attained for all geodesic balls, not necessarily the geodesic balls centered at p_0.

Another family of sharp geometric inequalities for static convex domains in hyperbolic space was obtained by using locally constrained inverse curvature flows [16].

Theorem 1.2 ([16]). *Let K be a smooth bounded domain in $\mathbb{H}^n$. Assume that ∂K is static convex and starshaped with respect to an interior point p_0 in K. For each $k=0,1,\ldots,n$, there holds*

$$\int_{\partial K}VH_k dA\geq n\left(\int_K Vd\mathrm{vol}\right)^{\frac{n-1-k}{n}}\left(|\mathbb{B}^n|^{\frac{2}{n}}+\left(\int_K Vd\mathrm{vol}\right)^{\frac{2}{n}}\right)^{\frac{k+1}{2}}, \tag{1.4}$$

where H_k is the normalized k-th mean curvature of the hypersurface ∂K. Equality holds in (1.4) if and only if K is a geodesic ball centered at p_0.

For $k=0$, the inequality (1.4) can be considered as a weighted isoperimetric inequality. It was first proved by Scheuer and Xia [33], and recently it was generalized to bounded domains with C^1 boundary by Li and Xu [24]. For $k=1$, the inequality (1.4) also holds for starshaped domains with mean convex boundary, see [33]. Moreover, the inequality (1.4) with $k=1$ also holds for merely static convex domains, i.e., p_0 is not necessarily an interior point of K, see [9, Thm. 4, case 4]. Using the well-known Minkowski's identity

$$n\int_K Vd\mathrm{vol}=\int_{\partial K}V_{,\nu}dA,$$

the inequality (1.4) with $k=1$ can be rewritten as

$$\int_{\partial K}H_1(\tilde{\kappa})dA\geq n|\mathbb{B}^n|^{\frac{2}{n}}\left(\int_K Vd\mathrm{vol}\right)^{\frac{n-2}{n}}, \tag{1.5}$$