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Examples of Ricci Solitons

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Abstract. In this survey paper, we discuss various examples of Ricci solitons and their constructions. Some open questions related to the rigidity and classification of Ricci solitons will be also discussed through those examples.

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1 Introduction

Ricci flow was introduced by Hamilton in 1982 [33], which is a parabolic equation for Riemannian metrics on a Riemannian manifold,

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g), \tag{1.1}$$

where $\mathrm{Ric}(g)$ is a Ricci tensor of g. As a class of singularity models, Ricci soliton plays a crucial role in the singularity analysis of Ricci flow ([34, 47]).

A Riemannian metric g on M^n is called a gradient Ricci soliton if there exists a smooth (potential) function f such that [†]

$$R_{ij} + \sigma g_{ij} = \nabla_i \nabla_j f, \tag{1.2}$$

where the constant σ can be normalized as $-1, 0, 1$ according to the type of Ricci solitons, namely, expanding, steady or shrinking, respectively. By a family of diffeomorphisms

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[†] In this paper, we always assume that Ricci solitons are gradient.

generated by the vector field $X=-\nabla f$ and suitable rescalings, a shrinking or steady Ricci soliton generates an ancient Ricci flow, which is defined for all $t\in(-\infty,0]$.

In $n=2$, it is known that there are only two examples of ancient non-flat Ricci solitons, namely, the cigar solution and the round sphere (cf. [15,23]). One can also find suitable potential functions on $\mathbb{R}^n$ so that the flat metric gives an expanding or shrinking solitons, which are known as Gaussian shrinking or expanding solitons.

Before we state the result in $n=3$, we recall

Definition 1.1 (κ-noncollapse). *(M,g) is an n-dimensional manifold which is κ-noncollapsed on scales r_0 if there exists some $\kappa>0$ such that for all $p\in M$*

$$\mathrm{vol}(B(p,r))\geq\kappa r^n,$$

whenever $|\mathrm{Rm}(q)|\leq r^{-2}$ $(r\leq r_0)$ for all $q\in B(p,r)$. (M,g) is κ-noncollapsed if it is κ-noncollapsed on all scales $r\leq\infty$.

A κ-noncollapsed ancient solution $g(t)$ of (1.1) is called a κ-solution if it has nonnegative curvature operator. In $n=3$, the condition is the same as the nonnegative sectional curvature. Thus by the Hamilton-Ivey curvature pinching estimate [36], any non-flat $3d$ κ-noncollapsed ancient solution is a κ-solution. Hence, we have following classification of $3d$ κ-noncollapsed ancient Ricci solitons.

Theorem 1.1 (Classification of $3d$ κ-noncollapsed ancient Ricci solitons [7,44]). *Let (M,g) be a $3d$ κ-noncollapsed Ricci solitons. Then the universal covering of (M,g) is one of following two cases:*

1) *Shrinking. It is either a round $3d$ sphere or a product of $2d$ round sphere and line;*

2) *Steady. It is the Byrant soliton up to a scale.*

Case 1) comes from a result of Perelman for nonnegative shrinking Ricci solitons by the splitting argument with the maximum principle [47] (also a general result of Munteanu-Wang for higher dimensional gradient shrinking Ricci solitons with non-negative curvature operator [44]). Case 2) comes from a result of Brendle [7], which solved a conjecture of Perelman about the uniqueness of $3d$ κ-noncollapsed Ricci soliton [47]. The Byrant soliton is a rotationally steady Ricci soliton with the maximal scalar curvature 1 [12].

Recently, Theorem 1.1 has been generalized for $3d$ ancient κ-solutions as follows [8, 10].

Theorem 1.2 (Classification of $3d$ ancient κ-solutions [8,10]). *Let (M,g) be a non-flat ancient κ-solution, Then*

1) *Noncompact. It is isometric to either shrinking cylinders or the Bryant soliton flow up to a scale;*

2) *Compact. It is either isometric to $3d$ shrinking quotient spheres or Perelman's solution.*

Case 1) in Theorem 1.2 is proved by Brendle [8], and also by Bamler-Kleiner [5]. Case 2) is proved by Brendle-Daskalopoulos-Sesum [10]. The Perelman's solution in Case 2) is a compact ancient κ-solution of type II on $\mathbb{S}^3$ [47].