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Examples of Ricci Solitons

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Abstract. In this survey paper, we discuss various examples of Ricci solitons and their constructions. Some open questions related to the rigidity and classification of Ricci solitons will be also discussed through those examples.

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1 Introduction

Ricci flow was introduced by Hamilton in 1982 [33], which is a parabolic equation for Riemannian metrics on a Riemannian manifold,

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g), \tag{1.1}$$

where $\mathrm{Ric}(g)$ is a Ricci tensor of g. As a class of singularity models, Ricci soliton plays a crucial role in the singularity analysis of Ricci flow ([34,47]).

A Riemannian metric g on M^n is called a gradient Ricci soliton if there exists a smooth (potential) function f such that [†]

$$R_{ij} + \sigma g_{ij} = \nabla_i \nabla_j f, \tag{1.2}$$

where the constant σ can be normalized as $-1,0,1$ according to the type of Ricci solitons, namely, expanding, steady or shrinking, respectively. By a family of diffeomorphisms

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†In this paper, we always assume that Ricci solitons are gradient.

generated by the vector field $X=-\nabla f$ and suitable rescalings, a shrinking or steady Ricci soliton generates an ancient Ricci flow, which is defined for all $t\in(-\infty,0]$.

In $n=2$, it is known that there are only two examples of ancient non-flat Ricci solitons, namely, the cigar solution and the round sphere (cf. [15,23]). One can also find suitable potential functions on $\mathbb{R}^n$ so that the flat metric gives an expanding or shrinking solitons, which are known as Gaussian shrinking or expanding solitons.

Before we state the result in $n=3$, we recall

Definition 1.1 (κ-noncollapse). *(M,g) is an n-dimensional manifold which is κ-noncollapsed on scales r_0 if there exists some $\kappa>0$ such that for all $p\in M$*

$$\mathrm{vol}(B(p,r))\geq \kappa r^n,$$

whenever $|\mathrm{Rm}(q)|\leq r^{-2}$ $(r\leq r_0)$ for all $q\in B(p,r)$. (M,g) is κ-noncollapsed if it is κ-noncollapsed on all scales $r\leq\infty$.

A κ-noncollapsed ancient solution $g(t)$ of (1.1) is called a κ-solution if it has nonnegative curvature operator. In $n=3$, the condition is the same as the nonnegative sectional curvature. Thus by the Hamilton-Ivey curvature pinching estimate [36], any non-flat $3d$ κ-noncollapsed ancient solution is a κ-solution. Hence, we have following classification of $3d$ κ-noncollapsed ancient Ricci solitons.

Theorem 1.1 (Classification of $3d$ κ-noncollapsed ancient Ricci solitons [7,44]). *Let (M,g) be a $3d$ κ-noncollapsed Ricci solitons. Then the universal covering of (M,g) is one of following two cases:*

1) *Shrinking. It is either a round $3d$ sphere or a product of $2d$ round sphere and line;*
2) *Steady. It is the Byrant soliton up to a scale.*

Case 1) comes from a result of Perelman for nonnegative shrinking Ricci solitons by the splitting argument with the maximum principle [47] (also a general result of Munteanu-Wang for higher dimensional gradient shrinking Ricci solitons with non-negative curvature operator [44]). Case 2) comes from a result of Brendle [7], which solved a conjecture of Perelman about the uniqueness of $3d$ κ-noncollapsed Ricci soliton [47]. The Byrant soliton is a rotationally steady Ricci soliton with the maximal scalar curvature 1 [12].

Recently, Theorem 1.1 has been generalized for $3d$ ancient κ-solutions as follows [8, 10].

Theorem 1.2 (Classification of $3d$ ancient κ-solutions [8,10]). *Let (M,g) be a non-flat ancient κ-solution, Then*

1) *Noncompact. It is isometric to either shrinking cylinders or the Bryant soliton flow up to a scale;*
2) *Compact. It is either isometric to $3d$ shrinking quotient spheres or Perelman's solution.*

Case 1) in Theorem 1.2 is proved by Brendle [8], and also by Bamler-Kleiner [5]. Case 2) is proved by Brendle-Daskalopoulos-Sesum [10]. The Perelman's solution in Case 2) is a compact ancient κ-solution of type II on $\mathbb{S}^3$ [47].

Although the classification of Ricci solitons in $n=3$, in particular with the κ-noncollaped conditions, is completely finished, the higher dimensional case is more complicated. There are many new examples of Ricci solitons with different topology and geometry. In this survey paper, we will discuss some open problems related to the rigidity and classification of Ricci solitons through various examples.

2 Examples and constructions

In this section, we present variant examples of Ricci solitons and as well as ancient solutions. These examples may help us to classify Ricci solitons under suitable geometric condition. We will discuss these examples according to their different construction method below.

2.1 Examples via ODE method

By reducing the soliton equation to solving a class of ODE, Bryant constructed a rotationally symmetric Ricci soliton on $\mathbb{R}^n$ as follows [12].

Let $g_{S^{n-1}}$ $(n\geq 3)$ denote the standard metric on the unit $(n-1)$-sphere S^{n-1}. We consider a warped product steady Ricci soliton on $\mathbb{R}^n$ by the form

$$g=dr^2+\phi(r)^2 g_{S^{n-1}}, \tag{2.1}$$

where $\phi(r)$ is smooth function for $r\in(0,+\infty)$ with asymptotic behavior $\phi(r)=r+O(1)$ near $r=0$. It was obtained in [12].

Example 2.1 (Bryant Ricci soliton)**.** There is a unique solution $\phi(r)$ depending only on the scalar curvature at the original such that the metric of form (2.1) gives an $O(n)$-invariant steady Ricci soliton, called the Bryant Ricci soliton.

One can show that ϕ satisfies the asymptotic behavior at infinity,

$$C^{-1}r^{\frac{1}{2}}\leq\phi(r)\leq Cr^{\frac{1}{2}},\phi'(r)=O\left(r^{-\frac{1}{2}}\right) \text{ and } \phi''(r)=O\left(r^{-\frac{3}{2}}\right). \tag{2.2}$$

Moreover, g has positive curvature operator with behavior of sectional curvature as

$$K_{\text{rad}}=O\left(r^{-2}\right),\quad K_{\text{sph}}=O\left(r^{-1}\right),$$

where K_{rad} denotes the sectional curvature of a plane passing through the radial vector $\frac{\partial}{\partial r}$ and K_{sph} denotes the sectional curvature of a plane perpendicular to $\frac{\partial}{\partial r}$. In particular, the scalar curvature has a linear decay. The volume behavior of g can also be obtained,

$$\text{Vol}(B(O,r))\approx\int_0^r s^{\frac{n-1}{2}}ds\approx r^{\frac{n+1}{2}}.$$

Under the κ-noncollapsed condition, it is known that the Bryant Ricci soliton is the only one up to scalings with nonnegative curvature operator and linear curvature decay by a result of Deng-Zhu [28]. Actually, we have

Theorem 2.1 ([28]). *Let (M,g) be an n-dimensional $(n\geq 4)$ noncompact κ-noncollapsed steady (gradient) Ricci soliton with nonnegative curvature operator. Then it is rotationally symmetric if its scalar curvaturte satisfies*

$$R(x)\leq \frac{C}{\rho(x)},$$

where $\rho(x)=\mathrm{dist}(x,p_0)$.

Other characters related to the rigidity of Bryant Ricci soliton have also been extensively studied in [27, 50–53], etc.

By the ODE method, Cao began to construct expanding and steady Kähler-Ricci solitons on $\mathbb{C}^n$ in 30 years ago [13]. Each ω of these metrics is $U(n)$-invariant which is determined by a Kähler potential $\phi(r)$ on $(\mathbb{C}^n;z)$ such that

$$\omega=\sqrt{-1}\partial\bar{\partial}\phi(r), \tag{2.3}$$

where $r^2=z\bar{z}=\sum=z^i\bar{z}^i$. Similar with the Bryant Ricci soliton, Cao's steady Kähler-Ricci soliton also depends only on the scalar curvature at the original and has positive bisectional (actually sectional) curvature. Moreover, the curvature has a linear decay when $n\geq 2$. ‡ Unfortunately, this steady Kähler-Ricci soliton is κ-collapsed. Actually, by a result of Deng-Zhu [25, 26], any κ-noncollapsed steady Kähler-Ricci soliton with non-negative bisectional curvature should be flat. However, up to now we do not know whether Cao's steady Kähler-Ricci soliton is the only one with positive bisectional curvature or not.

The Cao's method has been generalized to construct shrinking Kähler-Ricci solitons on holomorphic line bundles over $\mathbb{CP}^{n-1}$ by Feldman-Ilmanen-Knopf [32] as follows. Let L be the tautological holomorphic line bundle over $\mathbb{CP}^{n-1}$. Then for any integer k, the space $L^{-k}\backslash\mathbb{CP}^{n-1}$ is biholomorphic to $\mathbb{C}^n\setminus\{0\}/Z_k$. Thus any $U(n)$-invariant metric on L^{-k} can be regarded as an extending metric of form (2.3) on $\mathbb{C}^n\setminus\{0\}/\mathbb{Z}_k$.

Example 2.2 (FIK Ricci solitons). For any integer $0<k<n$ and positive number $p>0$, there is a shrinking Kähler-Ricci soliton on L^{-k}. Moreover, the tangent metric at infinity is a Kähler cone $\mathbb{C}^n\setminus\{0\}$ given by

$$h(z):=|z|^{2p-2}\left(\delta_{\alpha\bar{\beta}}+(p-1)|z|^{-2}\bar{z}^{\bar{\alpha}}z^{\beta}\right)dz^{\alpha}d\bar{z}^{\bar{\beta}}.$$

Up to now, all known non-trivial examples of complete shrinking Ricci solitons arise from Kähler geometry. In $n=2$, it was recently classified that there are only two kinds of complete shrinking Ricci solitons, one is of FIK Ricci solitons above, another is called the blow-up solution constructed by Bamler-Cifarelli-Conlon-Deruelle [4] (also see Example 2.8 below).

The Byrant's method can be generalized to construct steady Ricci solitons with multiple warped products. Appleton considered the following warped product metric on L^k

‡When $n=1$, the soliton is just the cigar solution with exponential decay curvature.

with positive integer k via the Hopf fibration $\pi: S^{2n-1}\to \mathbb{C}P^{n-1}$ [2],

$$g=ds^2+g_{a(s),b(s)}=ds^2+a(s)^2\sigma\otimes\sigma+b(s)^2\pi^* g_{\mathbb{C}P^{n-1}}, \tag{2.4}$$

where L is the tautological holomorphic line bundle over $\mathbb{CP}^{n-1}$ as in Example 2.2, and $g_{\mathbb{C}P^{n-1}}$ is the Fubini-Study metric on $\mathbb{C}P^{n-1}$.

Example 2.3 (Appleton's examples)**.** Let $k>n$. Then there exists a complete κ-noncollapsed steady Ricci soliton of form (2.4) on the completion of $\mathbb{R}_{>0}\times S^{2n-1}/\mathbb{Z}_k$ by adding $\mathbb{C}P^{n-1}$ at the origin, which is diffeomorphic to L^k. Moreover, we have

$$a\sim b\sim C\sqrt{s} \text{ as } s\to\infty$$

for a constant $C>0$ and the soliton is asymptotic to the quotient of the $2n$-dimensional Bryant soliton by $\mathbb{Z}_k$ at infinity. In particular, the curvature operator of g is positive outside of a compact subset and the scalar curvature has a linear decay at infinity.

2.2 Examples via the perturbation/deformation method

We first recall the Deruelle's construction for a class of expanding Ricci solitons which are asymptotic to cone metrics [29,30].

Definition 2.1. *Let (X^{n-1},g_X) be a compact Riemannian manifold and $(C(X),dr^2+r^2g_X)$ an n-dimensional cone metric with the link (X,g_X). An expanding Ricci soliton $(M^n,g;f)$ is asymptotically to a cone metric $(C(X),dr^2+r^2g_X)$ if there exists a compact $K\subset M$, a positive radius R, a diffeomorphism $\phi: M\setminus K\to C(X)\setminus B(o,R)$ such that*

$$\begin{aligned}
&\sup_{\partial B(o,r)}\left|\nabla^k\left(\phi_* g-g_{C(X)}\right)\right|_{g_{C(X)}}\le O(r^{-2-k}), \quad \forall k\in\mathbb{N},\\
&f\left(\phi^{-1}(r,x)\right)=\frac{r^2}{4}, \qquad\qquad \forall (r,x)\in C(X)\setminus B(o,R),\\
&f_k(r)=o(1), \quad \text{as } r\to+\infty, \qquad\qquad \forall k\ge 0.
\end{aligned}$$

Example 2.4 (Deruelle's expanding Ricci solitons)**.** Let (X^{n-1},g_X) be a smooth simply connected compact Riemmanian manifold such that the curvature operator $\mathrm{Rm}(g_X)\ge 1$. Then there exists a unique expanding Ricci soliton with nonnegative curvature operator, which is asymptotic to the cone metric $(C(X),dr^2+r^2g_X)$ with the link (X^{n-1},g_X).

Deruelle used a continuity method to construct the expanding Ricci solitons in Example 2.4. By Böhm-Wilking's result [6], one can start (X,g_X) via the normalized Ricci flow to obtain a family of metrics $(g(s))_{s\in[0,+\infty]}$ on X with constant volume and positive uniformly bounded curvature operator, which deform to the round sphere $\left(X,c^2g_{\mathbb{S}^{n-1}}\right)$, where $c^{n-1}=\mathrm{Vol}(X,g_X)\,/\,\mathrm{Vol}\left(\mathbb{S}^{n-1},g_{\mathbb{S}^{n-1}}\right)$. Thus, the initial cone metric $(C(X),dr^2+r^2g_X)$ can be connected to the cone metric $(C(\mathbb{S}^{n-1}),dr^2+(cr)^2g_{\mathbb{S}^{n-1}})$ by a family of cone metrics

$C(X_t)$, whose link $X_t=(X,g_t)$ has positive uniformly bounded curvature operator. Note that the cone metric $\left(C\left(\mathbb{S}^{n-1}\right)\right.$ is the Gauss metric, i.e., the Euclidean metric on $\mathbb{R}^n$, which can be also regarded as an expanding Ricci soliton. By the deformation method, Deruelle proved that there exist a family of expanding Ricci solitons with nonnegative curvature operator, each of which is asymptotic to the above cone metric on $C(X_t)$.

We also note that Conlon-Deruelle gave necessary and sufficient conditions for a Kähler equivariant resolution of Kähler cone to admit a unique asymptotically conical expanding gradient Kähler-Ricci soliton [19]. As a consequence, they constructed a class of expanding Kähler-Ricci solitons by solving certain complex Monge-Ampère equations.

The above deformation method has also been generalized to construct steady gradient Kähler-Ricci solitons by Cifarelli-Conlon-Deruelle [20] via the complex Monge-Ampère equation. Let (C_0,g_0) be a Calabi-Yau cone of complex dimension $n\geq 2$ with complex structure J_0, Calabi-Yau cone metric g_0, radial function r, and trivial canonical bundle. This means that

$$\omega_{g_0}=\sqrt{-1}\partial\bar{\partial}r^2$$

is a Ricci flat metric. Following Cao's ODE construction in [13], Cifarelli-Conlon-Deruelle showed that there exists a family of incomplete steady Kähler-Ricci solitons ω_a with same soliton vector field $X=2r\partial_r$ on C_0 [20, Proposition 2.18]. We note that ω_a has the same asymptotic behavior at infinity of C_0, it is not a cone metric in general.

Let $\pi:M\to C_0$ be a crepant resolution of C_0 with induced complex structure $J=\pi^*J_0$ such that the real holomorphic torus action on C_0 generated by $J_0r\partial_r$ extends to M and the holomorphic vector field $2r\partial_r$ on C_0 lifts to a real holomorphic vector field $X=\pi^*(2r\partial_r)$ on M. Set $t:=\log\left(r^2\right)$ and define the Kähler form

$$\hat{\omega}:=\frac{i}{2}\partial\bar{\partial}\left(\frac{nt^2}{2}\right)\ \text{ on } C_0.$$

Cifarelli-Conlon-Deruelle obtained.

Example 2.5 (CCD's steady Kähler-Ricci solitons)**.** In each Kähler class $\mathfrak{k}$ of M, up to the flow of X, there exists a unique complete steady Kähler-Ricci soliton $\omega\in\mathfrak{k}$ with soliton vector field X and with $\mathcal{L}_{JX}\omega=0$ such that for all $\varepsilon\in(0,1)$, there exist constants $C(i,j,\varepsilon)>0$ such that

$$\left|\widehat{\nabla}^i\mathcal{L}_X^{(j)}\left(\pi_*\omega-\hat{\omega}\right)\right|_{\hat{g}}\leq C(i,j,\varepsilon)t^{-\varepsilon-\frac{i}{2}-j},\qquad\forall\, i,j\in\mathbb{N}_0,$$

where $\hat{g}$ denotes the Kähler metric associated to $\hat{\omega}$ and $\widehat{\nabla}$ is the corresponding Levi-Civita connection.

In Examples 2.5, Cifarelli-Conlon-Deruelle further proved the Kähler-Ricci soliton ω converges at a polynomial rate to the Cao's steady Kähler-Ricci soliton ω_a at infinity of cone C_0. Thus the above examples also show that there is no uniqueness property for the steady Ricci solitons in general even with same topology and same metric behavior at infinity.

2.3 Examples via rescaling of Ricci flows

There are many examples of ancient solutions of Ricci flow obtained as the limit of a rescaled sequence of Ricci flows solutions. Perelman's solution is one of such famous examples on S^3 [48].

Consider a solution to the Ricci flow with initial metric on S^3 that looks like a long cylinder $S^2\times I$ (with radius one and length $L\gg 1$), with two spherical caps, smoothly attached to its boundary components. By [33], we know that after normalization the flow converges to the round S^3. Scale the initial metric and choose the time parameter in such a way that the flow starts at time $t_0=t_0(L)<0$, goes singular at $t=0$, and at $t=-1$ has the ratio of the maximal sectional curvature to the minimal one equal to $1+\epsilon$. Then after taking subsequence, the sequence of solutions converge to an ancient κ-solution on S^3 in the Cheeger-Gromov sense. Thus we get

Example 2.6 (Perelman's ancient solution)**.** There exists an ancient κ–solution of type II on S^3 for any $t\in(-\infty,0)$.

Recall that a compact ancient κ-solution $(N,h(t))$ $(t\in(-\infty,0])$ of type I means that it satisfies

$$\sup_{N\times(-\infty,0]}(-t)|R(x,t)|<\infty.$$

Otherwise, it is called type II, i.e., it satisfies

$$\sup_{M\times(-\infty,0]}(-t)|R(x,t)|=\infty.$$

Since any closed ancient κ-solution of Type I is a shrinking Ricci soliton ([17,46]) and so it is a family of quotient shrinking spheres in dimension 3 by Hamilton's theorem [33], Perelman's ancient solution must be type II. By the construction of Perelman's ancient solution, it has $Z_2\times O(3)$-symmetry, and seems more and more like a cylinder at the middle part as time goes to $-\infty$. Recently, the asymptotic behavior of Perelman's ancient solution has been proved by Angenent-Brendle-Daskalopoulos-Sesum as follows.

Theorem 2.2 ([1])**.** *Let $(S^3,g(t))$ be the Perelman's ancient solution. Then we can find a reference point $q\in S^3$ such that the following holds. Let $F(z,t)$ denote the radius of the sphere of symmetry in $(S^3,g(t))$ which has signed distance z from the reference point q. Then the profile $F(z,t)$ has the following asymptotic expansions:*

(i) Fix a large number L. Then, as $t\to-\infty$, we have

$$F(z,t)^2=-2t-\frac{z^2+2t}{2\log(-t)}+o\Big(\frac{(-t)}{\log(-t)}\Big)$$

for $|z|\le L\sqrt{-t}$.

(ii) Fix a small number $\theta>0$. Then as $t\to-\infty$, we have

$$F(z,t)^2=-2t-\frac{z^2}{2\log(-t)}+o(-t)$$

for $|z|\le 2\sqrt{1-\theta^2}\sqrt{(-t)\log(-t)}$.

(iii) The reference point q has distance $(2+o(1))\sqrt{(-t)\log(-t)}$ from each tip. The scalar curvature at each tip is given by $(1+o(1))\frac{\log(-t)}{(-t)}$. Finally, if we rescale the solution around one of the tips, then the rescaled solutions converge to the Bryant soliton as $t\to-\infty$.

The high dimensional construction of Perelman's ancient solution has also been studied. We refer the reader to [11,35].

Recently, Lai found another rescaling method to construct a new class of steady Ricci solitons with positive curvature operator for any dimension $n\ge 3$ [37,39]. Lai's construction is based on Deruelle's expanding Ricci solitons in Example 2.4.

Let $(M_i,g_i,p_i;f_i)$ be $\mathbb{Z}_2\times O(n-1)$-symmetric expanding solitons with nonnegative curvature operator, where p_i are the unique points fixed by the $\mathbb{Z}_2\times O(n-1)$-action. We denote the eigenvalues of the Ricci curvature at the point p_i as $\lambda_1(g)$ and $\lambda_2(g)=\cdots=\lambda_n(g)$, corresponding to the directions of edges and its orthogonal complement subspace, respectively. For any given $\alpha\in(0,1)$, one can find a sequence of expanding Ricci soliton with asymptotic volume ratio tending to zero satisfing $\frac{\lambda_1}{\lambda_2}(g_i)=\alpha$. By the compactness, after passing to a subsequence, Lai proved that $(M_i,g_i,p_i;f_i)$ converges to a $\mathbb{Z}_2\times SO(n-1)$-symmetric nonnegative curved steady Ricci solitons $(M_\alpha,g_\alpha,f_\alpha,p_\alpha)$. In particular, $\frac{\lambda_1}{\lambda_2}(g_\alpha)=\alpha$. Since we have $\frac{\lambda_1}{\lambda_2}(g)=0$ on $\mathbb{R}\times\mathrm{Bry}_{n-1}$ for $n\ge 4$ or $\mathbb{R}\times$Cigar for $n=3$, and $\frac{\lambda_i}{\lambda_2}(g)=1$ $(i=3,\dots,n)$ on n-dimensional Bryant soliton, the new family of steady Ricci solitons $(M_\alpha,g_\alpha,p_\alpha;f_\alpha)$ are different from the those $\mathbb{R}\times\mathrm{Bry}_{n-1}$ or $\mathbb{R}\times$Cigar.

These steady Ricci solitons are all κ-noncollapsed except $n=3$. In particular, she solved a conjecture of Hamilton for the existence of flying wings, each of which is neither the $3d$ Bryant soliton nor the $\mathbb{R}\times$Cigar. Thus flying wings ($n=3$) are all κ-collapsed by the classification theorem, Theorem 1.1. Without of confusion, we call Lai's examples of steady Ricci solitons by Lai's flying wings.

Example 2.7 (Lai's flying wings)**.** For any dimension $n\ge 3$, there exists a family of $\mathbb{Z}_2\times O(n-1)$-symmetric steady Ricci solitons with nonnegative curvature operator. Moreover, they are all κ-noncollapsed except $n=3$.

In the Kähler case, Chan-Conlon-Lai constructed a family of $U(1)\times U(n-1)$-invariant, but not $U(n)$-invariant, complete steady gradient Kähler-Ricci solitons with strictly positive curvature operator on real $(1,1)$-forms on $\mathbb{C}^n$ for $n\ge 3$, which can be regarded as Kähler generalization of Lai's flying wings [22].

By Perelman's result [47], the asymptotic volume rate of any nonnegative curved ancient solution must be zero. For $3d$ flying wings, the tangent cone at infinity (asymptotic cone) is a metric cone over the interval $[-\frac{\alpha}{2},\frac{\alpha}{2}]$ for some $\alpha\in[0,\pi]$. Let Γ_0 be the fixed point set of the $O(n-1)$-action. One can check Γ_0 is also the integral curve of the vector field ∇f. By the steady Ricci soliton equation, the scalar curvature is monotonically decreasing along Γ_0. Furthermore, Lai proved the relationship between the angle of asymptotic cone and the limit of scalar curvature along the edges satisfies

$$\lim_{s\to\infty} R(\Gamma_0(s)) = R(p)\sin^2(\frac{\alpha}{2}). \tag{2.5}$$

On the other hand, Lai proved the only $3d$ steady Ricci soliton whose asymptotic cone is a ray is the Bryant soliton [37]. Thus by (2.5) the scalar curvature does not decay to zero along the edges for the family of $3d$ flying wings.

For the higher dimensional case $n\geq 4$, we know fewer properties about Lai's examples besides the κ-noncollapsing property. Motivated from the mean curvature flow [14], Haslhofer conjectured that the asymptotic cone of any $4d$ nonnegatived κ-noncollapsed gradient Ricci soliton is either a ray or splits off a line [35]. Thus by Haslhofer's conjecture with the help of (2.5), the scalar curvature of $4d$ dimensional Lai's examples should decay to zero uniformly. It follows that the level set of the potential function f associated to the induced metric behaves more and more like a Perelman's ancient solution at infinity. Hence, there arises a natural conjecture for the optimal curvature decay of flying wings as follows.

Conjecture 2.1. *The scalar curvature of flying wings of dimension $n\geq 4$ satisfies*

$$\frac{C^{-1}}{\rho(\cdot)} \leq R(\cdot) \leq \frac{C\log\rho(\cdot)}{\rho(\cdot)}$$

for some constant $C>0$ depends on the soliton metric g.

Recently, Ma-Mahmoudian-Sesum verified the above conjecture in $4d$ by assuming a fast curvature decay condition [43].

It is also interesting to study the volume growth of steady Ricci solitons in Examples 2.7. By assuming that Conjecture 2.1 is true, we can prove

Proposition 2.1. *In $n=4$, the volume growth of flying wings satisfies*

$$\mathrm{Vol}(B(p,r)) \sim r^{\frac{5}{2}}\sqrt{\log r}, \qquad \forall\, r\gg 1.$$

Proof. We first estimate the volume of Perelman's ancient solution $g_{Pel}(t)$. Let F be the profile function in Theorem 2.2. We see

$$F(z,t)\leq 2\sqrt{(-t)}$$

for all z and $-t\gg 1$. Since $g(t)$ is a warped metric, by (iii) in Theorem 2.2, we get

$$\mathrm{Vol}(B_{g_{Pel}}(q,t)) \leq 4\pi(2\sqrt{(-t)})^2\sqrt{(-t)\log(-t)} = 16\pi(-t)^{\frac{3}{2}}\sqrt{\log(-t)} \tag{2.6}$$

for $-t\gg 1$. On the other hand, by the estimates of profile function F on the neck region in Theorem 2.2, we have

$$F(z,t)^2\geq -2t-\frac{2z^2}{3\log(-t)}$$

for $|z|\leq\sqrt{(-t)\log(-t)}$ and $-t\gg 1$. Thus we also get

$$\begin{aligned}\mathrm{Vol}(B_{g_{Pel}}(q,t))\geq & 2\pi\int_0^{\sqrt{(-t)\log(-t)}}\left(-2t-\frac{2z^2}{3\log(-t)}\right)dz\\ =&4\pi\left((-t)^{\frac{3}{2}}\sqrt{\log(-t)}-\frac{1}{9}(-t)^{\frac{3}{2}}\sqrt{\log(-t)}\right)\\ =&\frac{32}{9}\pi(-t)^{\frac{3}{2}}\sqrt{\log(-t)}.\end{aligned}\tag{2.7}$$

Now we estimate the volume growth of 4d flying wings under the assumption that the asymptotic behavior like the Perelman's ancient solution. Choose sufficiently large constant r_0, such that the level sets $\Sigma_r=\{x:f(x)=r\}$, where $r\geq r_0$, behave sufficiently like the Perelman's ancient solution under some uniform scaling. Then the estimates (2.6) and (2.7) hold for all $-t\geq r_0$. Thus for $r\gg 1$, by (2.6) we get

$$\begin{aligned}\mathrm{Vol}(B(p,r))\leq & 16\pi\int_0^{r_0}s^{\frac{3}{2}}\sqrt{\log s}ds+16\pi\int_{r_0}^{r}s^{\frac{3}{2}}\sqrt{\log s}ds\\ =&C+\frac{32}{5}\pi r^{\frac{5}{2}}\sqrt{\log r}-\frac{32}{5}\pi r_0^{\frac{5}{2}}\sqrt{\log r_0}-\frac{16}{5}\pi\int_{r_0}^{r}s^{\frac{3}{2}}\frac{1}{\sqrt{\log s}}ds.\end{aligned}\tag{2.8}$$

It follows

$$\mathrm{Vol}(B(p,r))\leq\frac{16}{5}\pi r^{\frac{5}{2}}\sqrt{\log r}-C.$$

Also, by (2.7) we obtain

$$\begin{aligned}\mathrm{Vol}(B(p,r))\geq & \frac{64}{45}\pi r^{\frac{5}{2}}\sqrt{\log r}-\frac{32}{45}\pi\int_{r_0}^{r}s^{\frac{3}{2}}\frac{1}{\sqrt{\log s}}ds-C\\ \geq & \frac{64}{45}\pi r^{\frac{5}{2}}\sqrt{\log r}-\frac{3}{5}\int_{r_0}^{r}s^{\frac{3}{2}}\sqrt{\log s}ds-C\\ \geq & \frac{64}{45}\pi r^{\frac{5}{2}}\sqrt{\log r}-\frac{3}{5}\mathrm{Vol}(B(p,r))-C.\end{aligned}$$

Hence,

$$\mathrm{Vol}(B(p,r))\geq\frac{8}{9}\pi r^{\frac{5}{2}}\sqrt{\log r}-C.$$

The proposition is proved. □

2.4 Examples from singularity models

By Hamilton's classification of singularity models [34] and Perelman's κ-noncollapsed result [47] for Ricci flow at finite time, the blow-up solution of type I singularities is ancient as well as the blow-up solution of type II singularities is eternal, i.e, the flow exists for any $t\in(-\infty,\infty)$. Actually, the first one is a shrinking Ricci soliton [16, 31, 45]. Recently, Bamler-Cifarelli-Conlon-Deruelle constructed such a Ricci soliton by studying Kähler-Ricci flow on the toric surface, the blow-up space $M=Bl_x(\mathbb{C}P^1\times\mathbb{C}P^1)$ of $\mathbb{C}P^1\times\mathbb{C}P^1$ at one point.

By choosing a suitable torus invariant initial metric on M, the Ricci flow $\omega(t)$ contracts the exceptional divisor and two boundary divisors at the maximal time T. It suffices to show that the blow-up solution with type I rescaling is a smooth ancient solution with uniformly bounded curvature. By a recent work of Bamler on the tangent Ricci flow associated to the Ricci flow [3], this blow-up solution is an ancient solution with isolated orbifold singularities. Thus the remaining step is to prove that this solution has uniformly bounded curvature and also removes the singularities.

Example 2.8 (BCCD's shrinking Ricci soliton [4])**.** There exists a torus invariant shrinking Kähler-Ricci soliton on the blow-up space $N=Bl_x(\mathbb{C}\times\mathbb{C}P^1)$ of $\mathbb{C}^1\times\mathbb{C}P^1$ at one point. This Ricci soliton can be realized as a type I singularity model of Ricci flow on $Bl_x(\mathbb{C}P^1\times\mathbb{C}P^1)$.

Since the toric surface N in Example 2.8 is different to the bundle space in Example 2.2, this shrinking Kähler-Ricci soliton on N is not the FIK Ricci soliton. On the other hand, the FIK Ricci soliton is the only non-flat shrinking Kähler-Ricci soliton on complex surfaces with curvature decay by a classification result of Conlon-Deruelle-Sun [21]. Thus the Ricci soliton in Example 2.8 can not have a curvature decay at infinity. Up to now, the Ricci soliton in Example 2.8 is the only known complete non-splitting shrinking Ricci soliton in $n=4$ with bounded curvature, but curvature does not decay to zero.

Type II Singularities of Ricci flow have also been found in the Kähler-Ricci flow on a class of Fano manifolds called as Fano G-manifolds or more general Fano horosymmetric manifolds in papers of Li-Tian-Zhu and Tian-Zhu [40, 49]. Actually, we have the following theorem.

Theorem 2.3 ([40,49])**.** *On any Fano G-manifold or more general Fano horosymmetric manifold, which does not admit any Kähler-Ricci soliton, the Kähler-Ricci flow will develop the type II Singularities whenever the initial metric is chosen in the canonical class.*

It is known that there are two Fano compactifications of $SO_4(\mathbb{C})$ and one Fano compactification of $Sp_4(\mathbb{C})$, on each of which does not admit a Kähler-Ricci soliton ([24, 41, 54]). Thus as an application of Theorem 2.3, we get

Example 2.9 (Type II Singularities on G-manifolds)**.** There are two Fano compactifications of $SO_4(\mathbb{C})$ and one Fano compactification of $Sp_4(\mathbb{C})$, each of which the Kähler-Ricci flow develops singularities of type II. As a consequence, the corresponding blow-up solutions of Ricci flow are all eternal.

It is not clear whether the corresponding blow-up solutions in Theorem 2.3 are exactly steady Kähler-Ricci solitons or not. § Also it is interesting to study whether the G-group action on those blow-up solutions are still preserved or not.

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§To authors' acknowledge, all known examples of blow-up solutions of type II are steady Ricci solitons up to date.

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