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Monotone Sequences of Metric Spaces with Compact Limits

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Abstract. In this paper, we consider a fixed metric space (possibly an oriented Riemannian manifold with boundary) with an increasing sequence of distance functions and a uniform upper bound on diameter. When the fixed space endowed with the pointwise limit of these distances is compact, then there is uniform and Gromov-Hausdorff (GH) convergence to this space. When the fixed metric space also has an integral current structure of uniformly bounded total mass (as is true for an oriented Riemannian manifold with boundary that has a uniform bound on total volume), we prove volume preserving intrinsic flat convergence to a subset of the GH limit whose closure is the whole GH limit. We provide a review of all notions and have a list of open questions at the end.

AMS subject classifications: 53C24

Key words: Metric spaces, Riemannian, Gromov-Hausdorff, intrinsic flat.

1 Introduction

Our goal is to teach the notions of Gromov-Hausdorff (GH) and Sormani-Wenger intrinsic flat (SWIF) convergence while proving a new theorem that has no assumptions on curvature. The notion of GH distance between metric spaces was first introduced by Edwards in [8] and deeply explored by Gromov in [10] and [9]. See Rong's 2010 survey [18] for many applications of GH convergence to sequences of Riemannian manifolds with curvature bounds.

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The notion of SWIF distance between integral current spaces was introduced by Sormani and Wenger in [22] applying the theory of Ambrosio-Kirchheim in [2] and work of Wenger in [23]. Volume preserving intrinsic flat ($\mathcal{VF}$) convergence was introduced by Portegies in [16]. See Sormani's survey [19] for many applications of SWIF convergence to sequences of Riemannian manifolds with lower bounds on their scalar curvature. See also papers by Allen, Bryden, Huang, Jauregui, Lakzian, Lee, Perales, Portegies, Sormani, and Wenger which prove SWIF and $\mathcal{VF}$ convergence theorems and present counterexamples [15,22,24] [5,6] [3,4,11] [1,12,13].

The work in this paper is inspired by a theorem in the appendix of [11] by Huang-Lee-Sormani (which applies to sequences with biLipschitz bounds on their distances) and a theorem within [4] by Allen-Perales-Sormani (which assumes only volume converging from above and distance from below but requires the limit space to be a compact smooth oriented Riemannian manifold). Here we will only assume the limit space is a compact metric space but we add the hypothesis that the distances are monotone increasing. Our result will be applied by Sormani-Tian-Yeung in [21] to prove SWIF and GH convergence to the extreme limits constructed by Sormani-Tian-Wang in [20].

Theorem 1.1. *Given a compact connected Riemannian manifold, (M^m,g_0), possibly with boundary, with a monotone increasing sequence of Riemannian metric tensors g_j such that*

$$g_j(V,V) \geq g_{j-1}(V,V) \qquad \forall V \in TM \tag{1.1}$$

with uniform bounded diameter,

$$\mathrm{diam}_{g_j}(M) \leq D_0 \qquad \forall j \in \mathbb{N}. \tag{1.2}$$

Then the induced length distance functions $d_j : M\times M \to [0,D_0]$ are monotone increasing and converge pointwise to a distance function, $d_\infty : M\times M \to [0,D_0]$ so that (M,d_∞) is a metric space.

If the metric space (M,d_∞) is a compact metric space, then $d_j \to d_\infty$ uniformly and we have Gromov-Hausdorff (GH) convergence,

$$(M,d_j) \xrightarrow{GH} (M,d_\infty). \tag{1.3}$$

If, in addition, M is an oriented manifold with uniform bounds on volume and boundary volume,

$$\mathrm{vol}_j(M) \leq V_0 \quad \textit{and} \quad \mathrm{vol}_j(\partial M) \leq A_0 \qquad \forall j \in \mathbb{N} \tag{1.4}$$

then we have volume preserving intrinsic flat ($\mathcal{VF}$) convergence

$$(M,d_j,[[M]]) \xrightarrow{\mathcal{VF}} (M_\infty,d_\infty,T_\infty), \tag{1.5}$$

where T_∞ is an integral current on (M,d_∞) such that "$T_\infty = [[M]]$ viewed as an integral current on (M,d_j)" and

$$M_\infty = \mathrm{set}_{d_\infty}(T_\infty) \subset M \textit{ with closure } \overline{M}_\infty = M. \tag{1.6}$$