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Monotone Sequences of Metric Spaces with Compact Limits

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In honor of Professor Xiaochun Rong on his seventieth birthday

Abstract. In this paper, we consider a fixed metric space (possibly an oriented Riemannian manifold with boundary) with an increasing sequence of distance functions and a uniform upper bound on diameter. When the fixed space endowed with the pointwise limit of these distances is compact, then there is uniform and Gromov-Hausdorff (GH) convergence to this space. When the fixed metric space also has an integral current structure of uniformly bounded total mass (as is true for an oriented Riemannian manifold with boundary that has a uniform bound on total volume), we prove volume preserving intrinsic flat convergence to a subset of the GH limit whose closure is the whole GH limit. We provide a review of all notions and have a list of open questions at the end.

AMS subject classifications: 53C24

Key words: Metric spaces, Riemannian, Gromov-Hausdorff, intrinsic flat.

1 Introduction

Our goal is to teach the notions of Gromov-Hausdorff (GH) and Sormani-Wenger intrinsic flat (SWIF) convergence while proving a new theorem that has no assumptions on curvature. The notion of GH distance between metric spaces was first introduced by Edwards in [8] and deeply explored by Gromov in [10] and [9]. See Rong's 2010 survey [18] for many applications of GH convergence to sequences of Riemannian manifolds with curvature bounds.

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The notion of SWIF distance between integral current spaces was introduced by Sormani and Wenger in [22] applying the theory of Ambrosio-Kirchheim in [2] and work of Wenger in [23]. Volume preserving intrinsic flat ($\mathcal{VF}$) convergence was introduced by Portegies in [16]. See Sormani's survey [19] for many applications of SWIF convergence to sequences of Riemannian manifolds with lower bounds on their scalar curvature. See also papers by Allen, Bryden, Huang, Jauregui, Lakzian, Lee, Perales, Portegies, Sormani, and Wenger which prove SWIF and $\mathcal{VF}$ convergence theorems and present counterexamples [15,22,24] [5,6] [3,4,11] [1,12,13].

The work in this paper is inspired by a theorem in the appendix of [11] by Huang-Lee-Sormani (which applies to sequences with biLipschitz bounds on their distances) and a theorem within [4] by Allen-Perales-Sormani (which assumes only volume converging from above and distance from below but requires the limit space to be a compact smooth oriented Riemannian manifold). Here we will only assume the limit space is a compact metric space but we add the hypothesis that the distances are monotone increasing. Our result will be applied by Sormani-Tian-Yeung in [21] to prove SWIF and GH convergence to the extreme limits constructed by Sormani-Tian-Wang in [20].

Theorem 1.1. *Given a compact connected Riemannian manifold, (M^m,g_0), possibly with boundary, with a monotone increasing sequence of Riemannian metric tensors g_j such that*

$$g_j(V,V) \geq g_{j-1}(V,V) \qquad \forall V \in TM \tag{1.1}$$

with uniform bounded diameter,

$$\mathrm{diam}_{g_j}(M) \leq D_0 \qquad \forall j \in \mathbb{N}. \tag{1.2}$$

Then the induced length distance functions $d_j : M\times M \to [0,D_0]$ are monotone increasing and converge pointwise to a distance function, $d_\infty : M\times M \to [0,D_0]$ so that (M,d_∞) is a metric space.

If the metric space (M,d_∞) is a compact metric space, then $d_j \to d_\infty$ uniformly and we have Gromov-Hausdorff (GH) convergence,

$$(M,d_j) \xrightarrow{GH} (M,d_\infty). \tag{1.3}$$

If, in addition, M is an oriented manifold with uniform bounds on volume and boundary volume,

$$\mathrm{vol}_j(M) \leq V_0 \quad \textit{and} \quad \mathrm{vol}_j(\partial M) \leq A_0 \qquad \forall j \in \mathbb{N} \tag{1.4}$$

then we have volume preserving intrinsic flat ($\mathcal{VF}$) convergence

$$(M,d_j,[[M]]) \xrightarrow{\mathcal{VF}} (M_\infty,d_\infty,T_\infty), \tag{1.5}$$

where T_∞ is an integral current on (M,d_∞) such that "$T_\infty = [[M]]$ viewed as an integral current on (M,d_j)" and

$$M_\infty = \mathrm{set}_{d_\infty}(T_\infty) \subset M \textit{ with closure } \overline{M}_\infty = M. \tag{1.6}$$

In fact, we prove more general statements that do not require M to be a manifold. See Theorem 3.1 within for GH convergence of a monotone sequence of metric spaces with a uniform upper bound on diameter whose distance functions converge pointwise to the distance function of a compact metric space. See Theorem 4.2 and Definition 4.1 within for $\mathcal{VF}$ and SWIF convergence. To keep the introduction simple, we do not state these results here.

Note that Sormani-Wenger already proved that any GH converging sequence satisfying (1.4) has a subsequence converging to a SWIF limit that is a possibly empty subset of the GH limit in [22]. In the Riemannian theorem above we use the monotonicity to force the whole sequence to converge and to force the closure of the SWIF limit to agree with the GH limit. We will see in Example 4.3 within that an increasing sequence can converge to a manifold with a cusp singularity, and since a cusp singularity is not included in the definition of an integral current space, this example has $M_\infty = \mathrm{set}(T_\infty) \neq M$.

We will prove this Riemannian theorem in stages as consequences of more general results concerning metric spaces and integral current spaces. We will review the various notions of convergence right before applying them, so that this article may be easily read by a novice. Note that GH, SWIF, and $\mathcal{VF}$ limit spaces are defined using distance functions and do not necessarily agree with the metric completions of limits spaces found by taking smooth convergence of Riemannian metric tensors away from singular sets (see work of Allen-Sormani [5]).

In Section 2 we prove a few basic lemmas about metric spaces with monotone increasing sequences of distance functions including a review of the Riemannian distance functions. In Lemma 2.3 we prove pointwise convergence of the monotone increasing sequence of distance functions with a uniform upper bound on diameter. In Example 2.1 we see that we need not have uniform convergence when the limit is noncompact.

In Section 3 we review the definition of Gromov-Hausdorff convergence (Definition 3.1). We then prove the GH and uniform convergence part of Theorem 1.1. In fact, we prove a more general theorem concerning a metric space with a monotone increasing sequence of distance functions that converge pointwise to a compact limit in Theorem 3.1. We show that the sequence converges uniformly and in the GH sense. We prove this theorem using different techniques than would have been used by Gromov in order to prepare for the SWIF convergence part of the paper. In particular, we construct a common metric space for the converging sequence that has special properties in Proposition 3.3 and Lemma 3.3. In Remark 3.1 we explain that Example 2.1 does not have a Gromov-Hausdorff limit.

In Section 4 we prove the SWIF convergence part of Theorem 1.1. In fact, we prove a more general SWIF convergence theorem concerning the convergence of an integral current space with increasing distance functions [Theorem 4.2]. We begin by reviewing De Giorgi's notion of a Lipschitz tuple [7], the notion of a Lipschitz chart, and Ambrosio-Kirchheim's notion of an integral current and weak convergence [2], proving lemmas about how these objects behave for monotone increasing sequences of distances. We review Sormani-Wenger's definition of an integral current space, including $(M,d,[[M]])$, and their definition of intrinsic flat distance. We then state Theorem 4.2 and apply it to complete the proof of Theorem 1.1 in Section 4.8. Finally we prove Theorem 4.2, first finding a converging subsequence and a limit current structure in Proposition 4.1 and completing the proof in Section 4.10 applying Proposition 4.2 to estimate the SWIF distance. In Section 5 we state open problems.

2 Pointwise convergence of monotone increasing distance functions

Definition 2.1. *A metric space, (X,d), is a collection of points, X, paired with a distance function $d: X\times X \to [0,\infty)$ which is definite,*

$$d(x,y)=0 \iff x=y, \tag{2.1}$$

symmetric,

$$d(x,y)=d(y,x) \qquad \forall x,y\in X, \tag{2.2}$$

and satisfies the triangle inequality,

$$d(x,y)\leq d(x,z)+d(z,y) \qquad \forall x,y,z\in X. \tag{2.3}$$

Here we consider metric spaces (X,d) of finite diameter,

$$\mathrm{diam}_d(X)=\sup\{d(x,y): x,y\in X\}<\infty. \tag{2.4}$$

We also need the following definition:

Definition 2.2. *We say that $d_j: X\times X\to[0,\infty)$ is a monotone increasing sequence of distance functions on X, if for all $j\in\mathbb{N}$ one has that (X,d_j) is a metric space and*

$$d_{j+1}(x,y)\geq d_j(x,y) \qquad \forall x,y\in X. \tag{2.5}$$

2.1 Riemannian distances

Recall that a Riemannian manifold, (M,g), has a natural metric space, (M,d_g), defined using the induced Riemannian distance as follows. The lengths of piecewise smooth curves, $C:[0,1]\to M$, are defined by

$$L_g(C)=\int_0^1 g(C'(t),C'(t))^{\frac{1}{2}}\,dt \tag{2.6}$$

and the Riemannian distance between points is defined by

$$d_g(x,y)=\inf\{L_g(C)\} \tag{2.7}$$

where the infimum is taken over all piecewise smooth curves, $C:[0,1]\to M$, such that $C(0)=x$ and $C(1)=y$. The diameter of (M,d_g) is then defined as

$$\mathrm{diam}_g(M)=\mathrm{diam}_{d_g}(M)=\sup\{d_g(x,y): x,y\in M\}. \tag{2.8}$$

Lemma 2.1. *If M is a Riemannian manifold with two Riemannian metric tensors g_a and g_b such that*

$$g_a(V,V) \le g_b(V,V) \qquad \forall V \in TM \tag{2.9}$$

then the corresponding induced Riemannian distances $d_a = d_{g_a}$ and $d_b = d_{g_b}$ satisfy

$$d_a(x,y) \le d_b(x,y) \qquad \forall x,y \in X. \tag{2.10}$$

The proof of Lemma 2.1 is an easy exercise applying 2.6 and 2.7.

Lemma 2.2. *Consider a fixed Riemannian manifold, (M,g_0), with a sequence of functions, $h_j : M \to [0,\infty)$, such that the gradients have magnitudes satisfying,*

$$|\nabla h_j|_{g_0} \le |\nabla h_{j+1}|_{g_0} \tag{2.11}$$

everywhere on M, and have the same direction,

$$\frac{\nabla h_j}{|\nabla h_j|_{g_0}} = \frac{\nabla h_{j+1}}{|\nabla h_{j+1}|_{g_0}} \tag{2.12}$$

whenever $|\nabla h_j|_{g_0} > 0$. Then the sequence of Riemannian metrics

$$g_j = g_0 + dh_j^2 \tag{2.13}$$

induced by the graphs of the functions, h_j, is a monotone increasing sequence of metric tensors, g_j as in as in (1.1), and defines a monotone increasing sequence of distance functions as in Definition 2.2.

Proof. By Lemma 2.1, we need only show (1.1). Recall that

$$g_j(V,V) = g_0(V,V) + dh_j^2(V,V) = g_0(V,V) + g_0(\nabla h_j, V)^2. \tag{2.14}$$

At points where $|\nabla h_j|_{g_0} = 0$, we thus have

$$g_j(V,V) = g_0(V,V) + 0 \le g_0(V,V) + dh_{j+1}^2(V,V) = g_{j+1}(V,V). \tag{2.15}$$

At points where $|\nabla h_j|_{g_0} > 0$, the gradients have the same direction so

$$\begin{aligned} g_j(V,V) &= g_0(V,V) + g_0(\nabla h_j, V)^2 \\ &= g_0(V,V) + g_0\left(\tfrac{\nabla h_j}{|\nabla h_j|}, V\right)^2 |\nabla h_j|^2 \\ &= g_0(V,V) + g_0\left(\tfrac{\nabla h_{j+1}}{|\nabla h_{j+1}|}, V\right)^2 |\nabla h_j|^2 \\ &\le g_0(V,V) + g_0\left(\tfrac{\nabla h_{j+1}}{|\nabla h_{j+1}|}, V\right)^2 |\nabla h_{j+1}|^2 \\ &= g_0(V,V) + dh_{j+1}^2(V,V) = g_{j+1}(V,V). \end{aligned} \tag{2.16}$$

This completes the proof that (1.1) holds at all points in M. □

2.2 Pointwise convergence

Lemma 2.3. *Suppose that $d_j : X \times X \to [0,\infty)$ is a monotone increasing sequence of distance functions on X as in Definition* 2.2 *and that there is a uniform upper bound on diameter,*

$$\operatorname{diam}_j(X) = \sup_{x,y\in X} d_j(x,y) \le D_0 \in (0,\infty) \tag{2.17}$$

then there is a limit distance function:

$$d_\infty(x,y) = \lim_{j\to\infty} d_j(x,y) = \sup_{j\in\mathbb{N}} d_j(x,y) \in [0,D_0] \tag{2.18}$$

and (X,d_∞) is a metric space with diameter $\operatorname{diam}_\infty(X) \le D_0$.

Proof. We know the pointwise limit function $d_\infty : X\times X \to [0,D_0]$ exists and satisfies (2.18) by the monotone convergence theorem for sequences of real numbers. We can confirm that d_∞ is positive definite as follows:

$$d_\infty(x,x) = \lim_{j\to\infty} d_j(x,x) = 0, \tag{2.19}$$

$$d_\infty(x,y) = 0 \implies d_j(x,y) \le 0 \implies x = y. \tag{2.20}$$

It is symmetric for all $x,y \in X$ by

$$d_\infty(x,y) = \lim_{j\to\infty} d_j(x,y) = \lim_{j\to\infty} d_j(y,x) = d_\infty(y,x) \qquad \forall x,y,\in X. \tag{2.21}$$

It satisfies the triangle inequality for any $x,y,z \in X$ by

$$\begin{aligned} d_\infty(x,y) &= \lim_{j\to\infty} d_j(x,y) \\ &\le \lim_{j\to\infty} (d_j(x,z) + d_j(z,y)) \\ &= \lim_{j\to\infty} d_j(x,z) + \lim_{j\to\infty} d_j(z,y) \\ &= d_\infty(x,z) + d_\infty(z,y) \qquad \forall x,y,z \in X. \end{aligned} \tag{2.22}$$

This completes the proof. □

2.3 Example without uniform convergence

Even if (X,d_j) is a monotone increasing sequence of compact metric spaces with a uniform upper bound on diameter, pointwise convergence does not imply uniform convergence. See Example 2.1 depicted in Fig. 2.3.

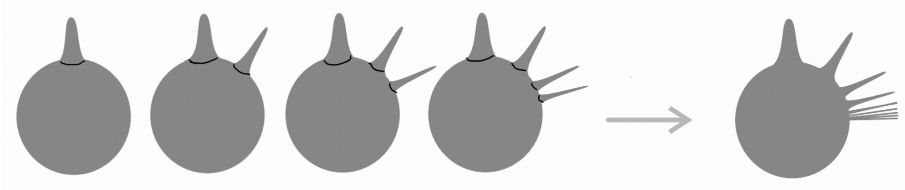

Figure 1: This figure depicts Example 2.1.

Example 2.1. Here we adapt an example from Sormani-Wenger [22] as in Fig. 2.3. Consider a sequence of distinct points $p_j \to p_\infty$ lying on the equator in the standard round sphere $(\mathbb{S}^2, g_{\mathbb{S}^2})$. Let $r_j > 0$ be decreasing radii such that the balls $B_{g_{\mathbb{S}^2}}(p_j, r_j)$ are disjoint and do not contain p_∞. We define a sequence of metric tensors

$$g_j = g_{\mathbb{S}^2} + dh_j^2 \tag{2.23}$$

inductively starting with constant $h_0 = 0$ and taking

$$h_{j+1}(x) = \begin{cases} h_j(x) & \text{if } x \in \mathbb{S}^2 \setminus B_{g_{\mathbb{S}^2}}(p_{j+1}, r_{j+1}) \\ H(d_{\mathbb{S}^2}(p_{j+1}, x)/r_{j+1}) & \text{if } x \in B_{g_{\mathbb{S}^2}}(p_{j+1}, r_{j+1}) \end{cases} \tag{2.24}$$

where $H: (-1,1) \to [0, H_0]$ is a fixed smooth even function of compact support such that $H(0) = H_0 > 0$ and $H'(r) \le 0$ for $r \ge 0$.

This implies that, for x outside of the disjoint balls about p_0 to p_i, we have

$$h_{i+1}(x) = h_i(x) = \ldots = h_0(x) = 0. \tag{2.25}$$

Since the balls are disjoint, for all $x \in B_{g_{\mathbb{S}^2}}(p_j, r_j)$, x is not in the previous balls, so $h_j(x) = 0$, so $\nabla h_j = 0$. Elsewhere, by (2.24) we have $h_j = h_{j+1}$ so $\nabla h_j = \nabla h_{j+1}$. Thus we can apply Lemma 2.2 to see that we have a monotone increasing sequence of metric tensors.

Note that we have added increasingly thin wells in place of each of the balls. For all $i \le j$, the g_j radial distance from p_i to any $q_i \in \partial B_{g_{\mathbb{S}^2}}(p_j, r_j)$,

$$d_{g_j}(p_i, q_i) = \int_0^{r_i} \sqrt{1 + (H'(r/r_i)(1/r_i))^2}\, dr. \tag{2.26}$$

Since $a \le \sqrt{1+a^2} \le 1+a$ we have

$$H_0 \le d_{g_j}(p_i, q) \le r_i + H_0 \qquad \forall i \le j. \tag{2.27}$$

Since any point in $(\mathbb{S}^2, g_j)$ outside of the wells can be joined to a pole traveling a distance $\le \pi/2$, and poles are a distance π apart, we can apply the triangle inequality to see that

$$\operatorname{diam}_{g_j}(\mathbb{S}^2) \le (H_0 + r_1 + \pi/2) + (H_0 + r_1 + \pi/2) + \pi. \tag{2.28}$$

This uniform upper bound and Lemma 2.3 implies there is a pointwise limit, $d_j \to d_\infty$. Taking the limit of (2.27) we have,

$$H_0 \le d_\infty(p_i, p_\infty) \qquad \forall i \in \mathbb{N}. \tag{2.29}$$

However, for $i > j$ sufficiently large

$$d_{g_j}(p_i, p_\infty) \le d_{\mathbb{S}^2}(p_i, p_\infty) < H_0/2 \tag{2.30}$$

so there is no uniform convergence. In fact, the limit distance function d_∞ has a sequence of disjoint g_j balls centered at the p_j of radius H_0, so $(\mathbb{S}^2, d_\infty)$ is not compact.

3 GH convergence to compact limits

Recall the following definition of the Gromov-Hausdorff distance first introduced by Edwards in [8] and then rediscovered and studied extensively by Gromov in [10].

Definition 3.1. *Given a pair of metric spaces, (X_a, d_a) and (X_b, d_b), the Gromov-Hausdorff distance between them is*

$$d_{GH}((X_a, d_a), (X_b, d_b)) = \inf\{d_H^Z(f_a(X_a), f_b(X_b))\} \tag{3.1}$$

where the infimum is taken over all common metric spaces, (Z, d_Z), and over all distance preserving maps,

$$f_a : (X_a, d_a) \to (Z, d_Z) \quad \text{and} \quad f_b : (X_b, d_b) \to (Z, d_Z). \tag{3.2}$$

Here d_H^Z denotes the Hausdorff distance,

$$d_H^Z(A, B) = \inf\{R : A \subset T_R(B) \text{ and } B \subset T_R(A)\} \tag{3.3}$$

between subsets $A, B \subset Z$ which is defined using tubular neighborhoods of a given radius R,

$$T_R(A) = \{z \in Z : \exists p \in A s.t. d_Z(p, z) < R\}. \tag{3.4}$$

In this section, we prove the following simple theorem:

Theorem 3.1. *If (X, d_j) is a monotone increasing sequence as in Definition 2.2 with a uniform upper bound D_0 on diameter, and if d_j converges pointwise to d_∞ as in Lemma 2.3, and if (X, d_∞) is a compact metric space, then d_j converge uniformly to d_∞ and we have Gromov-Hausdorff convergence as well.*

Although this GH Convergence theorem can be proven quite easily, we will prove it through a sequence of lemmas and propositions that we will apply again to prove SWIF convergence later.

3.1 Distance on the product space

We introduce the following standard definition:

Definition 3.2. *Given a pair of metric spaces, (X_a,d_a) and (X_b,d_b), we define the taxi product metric space $(X_a\times X_b,d_{sum,a,b})$ where*

$$d_{sum,a,b}((x_1,y_1),(x_2,y_2)) = d_a(x_1,x_2)+d_b(y_1,y_2) \tag{3.5}$$

for any $x_1,x_2\in X_a$ and any $y_1,y_2\in X_b$.

Lemma 3.1. *If (X,d_a) and (X,d_b) are compact then $(X\times X,d_{sum,a,b})$ is also compact.*

The proof is an exercise for the reader.

Lemma 3.2. *Given two metric spaces (X,d_a) and (X,d_b), if*

$$d_a(x,y)\le d_b(x,y) \qquad \forall x,y\in X \tag{3.6}$$

then the function $d_a: X\times X\to[0,\infty)$ is 1-Lipschitz with respect to the distance $d_{sum,b,b}$ on $X\times X$ given in Definition 3.2. That is,

$$|d_a(x_1,y_1)-d_a(x_2,y_2)|\le d_{sum,b,b}((x_1,y_1),(x_2,y_2)) \qquad \forall (x_i,y_i)\in X\times X. \tag{3.7}$$

Proof. By the triangle inequality and then by (3.6) we have

$$\begin{aligned}|d_a(x_1,y_1)-d_a(x_2,y_2)| &\le d_a(x_1,x_2)+d_a(y_1,y_2)\\ &\le d_b(x_1,x_2)+d_b(y_1,y_2)\end{aligned} \tag{3.8}$$

which gives our claim by (3.5) □

3.2 Uniform convergence

Proposition 3.1. *Under the hypotheses of Theorem 3.1, the distance functions, d_j, converge uniformly to d_∞ on $X\times X$.*

Proof. By Lemma 2.3, we have $d_j\le d_\infty$ as functions on $X\times X$. By Lemma 3.2, the maps $d_j: X\times X\to[0,D_0]$ are bounded 1-Lipschitz functions on the metric space $(X\times X,d_{sum,\infty,\infty})$, which is compact by hypothesis and Lemma 3.1.

By the Arzela-Ascoli Theorem, there is a subsequence, $d_{j_k}: X\times X\to[0,D_0]$, which converges uniformly to a limit function. However, we already have a pointwise limit, so that limit function equals $d_\infty: X\times X\to[0,D_0]$. That is, $\forall\epsilon>0$, $\exists N_\epsilon\in\mathbb{N}$, $s.t.\ \forall k\ge N_\epsilon$

$$d_\infty(x,y)-\epsilon< d_{j_k}(x,y)< d_\infty(x,y)+\epsilon \qquad \forall x,y\in X\times X. \tag{3.9}$$

By the assumption of monotonicity, for all $j\ge j_k\ge j_{N_\epsilon}$,

$$d_\infty(x,y)-\epsilon< d_{j_k}(x,y)\le d_j(x,y)\le d_\infty(x,y) \qquad \forall x,y\in X\times X, \tag{3.10}$$

showing that the whole sequence d_j uniformly converges to d_∞. □

3.3 Constructing a common metric space

Recall that the definition of the Gromov-Hausdorff distance involves a common metric space Z and distance preserving maps Definition 3.1. The following lemma is not part of the original proof by Gromov that uniform convergence implies Gromov-Hausdorff convergence. Here we create a different common metric space Z that allows us to prove intrinsic flat convergence later as well. It is a simplification of the Z constructed by Allen-Perales-Sormani in [4].

Proposition 3.2. *Given two metric spaces (X,d_a) and (X,d_b) and an $\epsilon > 0$ such that*

$$d_b(x,y) - \epsilon \le d_a(x,y) \le d_b(x,y) \qquad \forall x,y \in X, \tag{3.11}$$

then there exists a metric space

$$(Z_{a,b} = X \times [0,h], d_{Z_{a,b}}), \tag{3.12}$$

where $h = \epsilon/2$, with the distance between $z_1 = (x_1,t_1)$ and $z_2 = (x_2,t_2)$ defined by

$$d_{Z_{a,b}}(z_1,z_2) = \min\{d_{taxi_b}(z_1,z_2), (h-t_1)+(h-t_2)+d_a(x_1,x_2)\}, \tag{3.13}$$

where

$$d_{taxi_b}(z_1,z_2) = d_b(x_1,x_2) + |t_1 - t_2|, \tag{3.14}$$

so that the identity map,

$$id_{a,b} : (Z_{a,b}, d_{taxi_b}) \to (Z_{a,b}, d_{Z_{a,b}}) \tag{3.15}$$

is 1-Lipschitz and there are distance preserving maps:

$$f_a = f_{a,(a,b)} : (X,d_a) \to Z_{a,b} \quad s.t.\ f_a(x) = (x,h), \tag{3.16}$$

$$f_b = f_{b,(a,b)} : (X,d_b) \to Z_{a,b} \quad s.t.\ f_b(x) = (x,0), \tag{3.17}$$

so that

$$d_{GH}((X,d_a),(X,d_b)) \le d_H^{Z_{a,b}}(f_a(X), f_b(X)) \le h. \tag{3.18}$$

The intuition behind this construction is that we are taking the taxi product of (X,d_b) with the interval $([0,h], d_{\mathbb{R}})$ and then gluing (X,d_a) to this taxi product at $t=h$ which gives possibly shorter distances. See Fig. 2.

Proof. First we confirm that $d_{Z_{a,b}}$ is a metric. It is clearly definite and symmetric. To see that it satisfies the triangle inequality we have several cases. Let $z_i = (x_i,t_i) \in Z_{a,b}$, $i = 1,2,3$ and

$$D_{13} = d_{Z_{a,b}}(z_1,z_2) + d_{Z_{a,b}}(z_2,z_3). \tag{3.19}$$

Case I: Assume that $d_{Z_{a,b}}(z_1,z_2) = d_{taxi_b}(z_1,z_2)$ and $d_{Z_{a,b}}(z_2,z_3) = d_{taxi_b}(z_2,z_3)$. Then since d_{taxi_b} is a distance function, and thus satisfies the triangle inequality, and by definition of $d_{Z_{a,b}}$, we get

$$D_{13} = d_{taxi_b}(z_1,z_2) + d_{taxi_b}(z_2,z_3) \ge d_{taxi_b}(z_1,z_3) \ge d_{Z_{a,b}}(z_1,z_3). \tag{3.20}$$

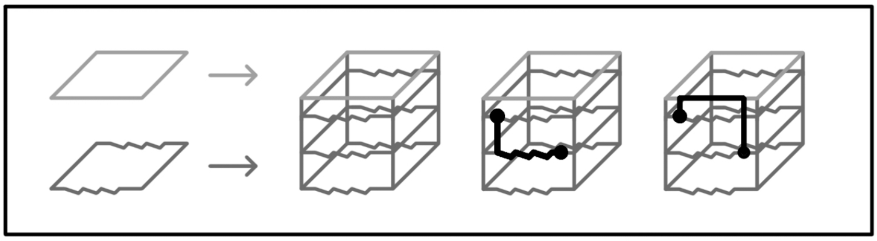

Figure 2: On the left we see (X_a,d_a) above (X_b,d_b) mapping into $(Z_{a,b},d_{Z_{a,b}})$ and on the right we see two paths between points in $Z_{a,b}$ where the first achieves d_{taxi_b} and the second takes a shortcut through the image of X_a. The shorter of the two paths achieves the minimum in the definition of $d_{Z_{a,b}}$.

Case II: Assume that we have $d_{Z_{a,b}}(z_1,z_2)=d_{taxi_b}(z_1,z_2)$ and

$$d_{Z_{a,b}}(z_2,z_3)=(h-t_2)+(h-t_3)+d_a(x_2,x_3). \tag{3.21}$$

Then by $d_b \geq d_a$, triangle inequality in $\mathbb{R}$ and for d_a, we get

$$\begin{aligned} D_{13}&=d_{taxi_b}(z_1,z_2)+(h-t_2)+(h-t_3)+d_a(x_2,x_3)\\ &\geq d_a(x_1,x_2)+|t_1-t_2|+(h-t_2)+(h-t_3)+d_a(x_2,x_3)\\ &\geq d_a(x_1,x_2)+(h-t_1)+(h-t_3)+d_a(x_2,x_3)\\ &\geq d_a(x_1,x_3)+(h-t_1)+(h-t_3)\geq d_{Z_{a,b}}(z_1,z_3). \end{aligned} \tag{3.22}$$

Case III: Assume that

$$d_{Z_{a,b}}(z_1,z_2)=(h-t_1)+(h-t_2)+d_a(x_1,x_2), \tag{3.23}$$

$$d_{Z_{a,b}}(z_2,z_3)=(h-t_2)+(h-t_3)+d_a(x_2,x_3). \tag{3.24}$$

Then, since $h-t_2\geq 0$ and triangle inequality for d_a, we get

$$\begin{aligned} D_{13}=&(h-t_1)+(h-t_2)+d_a(x_1,x_2)\\ &\quad+(h-t_2)+(h-t_3)+d_a(x_2,x_3)\\ \geq&(h-t_1)+d_a(x_1,x_2)+(h-t_3)+d_a(x_2,x_3)\\ \geq&d_a(x_1,x_3)+(h-t_1)+(h-t_3)\\ \geq&d_{Z_{a,b}}(z_1,z_3), \end{aligned} \tag{3.25}$$

which completes the proof of the triangle inequality on $(Z_{a,b},d_{Z_{a,b}})$.

We have the claimed 1-Lipschitz identity map, because $d_{Z_{a,b}}\leq d_{taxi_b}$. We have the claims that f_a and f_b are distance preserving maps as follows. By the definition of $d_{Z_{a,b}}$, f_a and $d_b\geq d_a$,

$$d_{Z_{a,b}}(f_a(x),f_a(y))=d_{Z_{a,b}}((x,h),(y,h))=\min\{d_b(x,y),d_a(x,y)\}=d_a(x,y). \tag{3.26}$$

By definition of $d_{Z_{a,b}}$, f_b, (3.11) and $h=\epsilon/2$ we have

$$d_{Z_{a,b}}(f_b(x),f_b(y))=\min\{d_b(x,y),2h+d_a(x,y)\}=d_b(x,y). \tag{3.27}$$

Furthermore, by the definition of $d_{Z_{a,b}}$, for any $x\in X$, it holds

$$d_{Z_{a,b}}((x,0),(x,h))=h, \tag{3.28}$$

which implies that for any $R>h$,

$$f_b(X)\subset T_R(f_a(X)) \quad \text{and} \quad f_a(X)\subset T_R(f_b(X)), \tag{3.29}$$

so the Hausdorff distance $d_H^Z(f_a(X),f_b(X))\le h$ which gives our final claim. □

3.4 Proof of Theorem 3.1

GH Convergence is now easy to prove:

Proof. First we apply Proposition 3.1 to get uniform convergence of d_j to d_∞ from below. Then we apply Proposition 3.2 to (X,d_j) and (X,d_∞) to see that

$$d_{GH}((X,d_j),(X,d_\infty))<\epsilon_j\to 0.$$

□

Remark 3.1. Note that in the proof of Theorem 3.1 it was essential that we assumed (X,d_∞) is compact, as seen in Example 2.1. This example has the monotonicity but no Gromov-Hausdorff limit. In fact, Gromov proved that if a sequence of compact metric spaces has a GH limit then the entire sequence can be embedded via distance preserving maps into a common metric space, (Z,d_Z) [9]. However, the Riemannian manifolds in Example 2.1 have an unbounded number of disjoint balls of radius 1, so this is impossible.

3.5 A common compact metric space for the whole sequence

Gromov proved in [9] that an entire GH-converging sequence of metric spaces can be embedded with distance preserving maps into a common compact metric space Z along with the limit so that the sequence of images converges in the Hausdorff sense inside Z to the image of the limit. Here we construct a specific Z which we will use later to prove SWIF convergence and Theorem 1.1. See Fig. 3.

Proposition 3.3. *If (X,d_j) is uniformly converging to compact (X,d_∞)*

$$d_\infty(x,y)-\epsilon_j<d_j(x,y)\le d_\infty(x,y) \qquad \forall x,y\in X \tag{3.30}$$

with ϵ_j decreasing to 0 then taking Z_j of Proposition 3.2, we define

$$Z=\bigsqcup_{j=1}^{\infty} Z_j|_\sim \text{ where } Z_j=Z_{j,\infty}=X_j\times[0,h_j], \tag{3.31}$$

where $(x,0)\in Z_j$ *is identified with* $(x,0)\in Z_k$ *for all* $x\in X$ *and all* $j,k\in\mathbb{N}$*. Then* $d_Z:Z\times Z\to[0,\infty)$ *given by*

$$d_Z(z_j,z_k)=\inf\{d_{Z_j}(z_j,(x,0))+d_{Z_k}((x,0),z_k)\,|\,x\in X\} \tag{3.32}$$

for

$$z_j=(x_j,t_j)\in Z_j \quad \textit{and} \quad z_k=(x_k,t_k)\in Z_k \qquad j\neq k, \tag{3.33}$$

and, otherwise,

$$d_Z(z_j,z_k)=d_{Z_j}(z_j,z_k), \tag{3.34}$$

so the inclusion maps,

$$\zeta_j:(Z_j,d_{Z_j})\to(Z,d_Z), \tag{3.35}$$

are distance preserving. In addition, taking

$$f_j=f_{j,(j,\infty)}:(X,d_j)\to Z_j \quad \textit{and} \quad f_{\infty,(j,\infty)}:(X,d_\infty)\to Z_j \tag{3.36}$$

from Proposition 3.2*, we have distance preserving maps:*

$$\chi_j=\zeta_j\circ f_j:(X,d_j)\to(Z,d_Z), \tag{3.37}$$

$$\chi_\infty=\zeta_j\circ f_{\infty,(j,\infty)}:(X,d_\infty)\to(Z,d_Z), \tag{3.38}$$

where $\chi_\infty(x)=\zeta_j(x,0)$ *does not depend on* j *and is an isometry onto,*

$$Z_0=\chi_\infty(X)=\zeta_j(f_{\infty,(j,\infty)}(X))\subset Z. \tag{3.39}$$

Furthermore, we have

$$d_Z(\chi_j(x),\chi_\infty(x))=d_{Z_j}((x,h_j),(x,0))=h_j \tag{3.40}$$

and

$$d_H^Z(\chi_j(X)),\chi_\infty(X))\le h_j. \tag{3.41}$$

Finally (Z,d_Z) *is compact.*

Intuitively, we have glued the Z_j together like the pages of a book, $\zeta_j:Z_j\to Z$, along the spine, Z_0, as in Fig. 3.

Proof. It is clear that d_Z is symmetric and positive definite. To see that it satisfies the triangle inequality we have several cases. Let $z_{j_i}=(x_i,t_i)\in Z_{j_i}$, $j_i\in\mathbb{N}$ and $i=1,2,3$,

$$D_{13}=d_Z(z_{j_1},z_{j_2})+d_Z(z_{j_2},z_{j_3}). \tag{3.42}$$

Case I: Assume that $j_1=j_2$ and $j_2=j_3$. In this case, $d_Z=d_{Z_{j_i}}$ which satisfies the triangle inequality since $d_{Z_{j_i}}$ is a distance function.

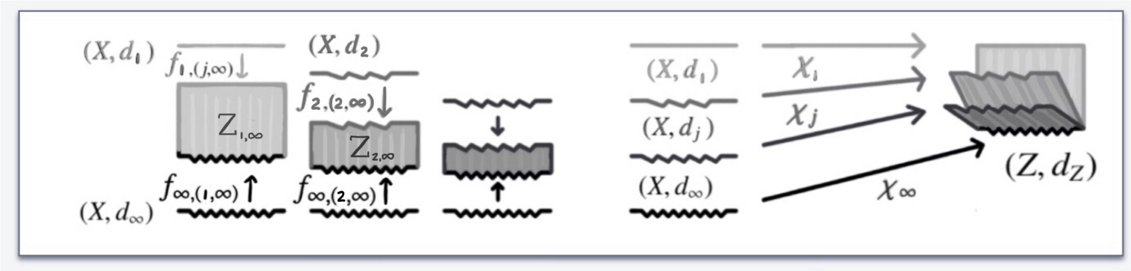

Figure 3: On the left, we see the sequence (X,d_j) above with downward maps $f_{j,(j,\infty)}:(X,d_j)\to(Z_{j,\infty},d_{Z_{j,\infty}})$ and upward maps $f_{\infty,(j,\infty)}:(X,d_\infty)\to(Z_{j,\infty},d_{Z_{j,\infty}})$ as constructed in Proposition 3.2 and depicted in Fig. 2. On the right, we glue together all the $(Z_{j,\infty},d_{Z_{j,\infty}})$ to create (Z,d_Z) as in Proposition 3.3 with maps $\chi_j:X_j\to Z$ and $\chi_\infty:X_j\to Z$.

Case II: Assume that $j_1=j_2$ and $j_2\neq j_3$. By $j_1=j_2$, the triangle inequality for $d_{Z_{j_2}}$ and definition of d_Z we get

$$\begin{aligned}D_{13}=&d_{Z_{j_2}}(z_{j_1},z_{j_2})+\inf_{x\in X}\{d_{Z_{j_2}}(z_{j_2},(x,0))+d_{Z_{j_3}}((x,0),z_{j_3})\}\\=&\inf_{x\in X}\{d_{Z_{j_2}}(z_{j_1},z_{j_2})+d_{Z_{j_2}}(z_{j_2},(x,0))+d_{Z_{j_3}}((x,0),z_{j_3})\}\\\geq&\inf_{x\in X}\{d_{Z_{j_2}}(z_{j_1},(x,0))+d_{Z_{j_3}}((x,0),z_{j_3})\}=d_Z(z_{j_1},z_{j_3}).\end{aligned}\tag{3.43}$$

Case III: Assume that $j_1\neq j_2$ and $j_2\neq j_3$. Take $x,x'\in X$, using the triangle inequality for $d_{Z_{j_2}}$ and $d_{Z_{j_1}}$, and the definition of d_Z we get

$$\begin{aligned}&d_{Z_{j_1}}(z_{j_1},(x,0))+d_{Z_{j_2}}((x,0),z_{j_2})+d_{Z_{j_2}}(z_{j_2},(x',0))+d_{Z_{j_3}}((x',0),z_{j_3})\\\geq&d_{Z_{j_1}}(z_{j_1},(x,0))+d_{Z_{j_2}}((x,0),(x',0))+d_{Z_{j_3}}((x',0),z_{j_3})\\\geq&d_{Z_{j_1}}(z_{j_1},(x,0))+d_{Z_{j_1}}((x,0),(x',0))+d_{Z_{j_3}}((x',0),z_{j_3})\\\geq&d_{Z_{j_1}}(z_{j_1},(x',0))+d_{Z_{j_3}}((x',0),z_{j_3})\geq d_Z(z_{j_1},z_{j_3}).\end{aligned}\tag{3.44}$$

Taking the infimum over $x,x'\in X$ in the previous expression we obtain $D_{13}\geq d_Z(z_{j_1},z_{j_3})$. This concludes the proof that (Z,d_Z) is a metric space.

The inclusion maps ζ_j are distance preserving by their definition and the definition of d_Z. Since by Proposition 3.2 $f_j=f_{j,(j,\infty)}$ and $f_{\infty,(j,\infty)}$ are distance preserving maps, then the compositions $\chi_j=\zeta_j\circ f_j$ and $\chi_\infty=\zeta_j\circ f_{\infty,(j,\infty)}$ are distance preserving maps. Since we are identifying points of the form $(x,0)\in Z_j$ with points $(x,0)\in Z_k$ for all $x\in X$, then $\chi_\infty(x)=\zeta_j(x,0)$ does not depend on j. Furthermore, χ_∞ is an isometry onto $Z_0=\chi_\infty(X)$.

Since χ_j and χ_∞ are distance preserving maps and by (3.18) in Proposition 3.2 we have

$$d_Z(\chi_j(x),\chi_\infty(x))\leq d_{Z_j}((x,h_j),(x,0))\leq h_j\tag{3.45}$$

so we have (3.40)-(3.41).

Finally, we check that (Z,d_Z) is compact. Given any sequence $z_i \in Z$,

$$\exists j_i \in \mathbb{N}\, \exists (x_i,t_i) \in Z_{j_i} = X \times [0,h_{j_i}] \text{ such that } z_i = \zeta_{j_i}(x_i,t_i). \tag{3.46}$$

Since (X,d_∞) is compact, a subsequence $x_i \to x_\infty$. Since $[0,1]$ is compact, a further subsequence can be taken such that $t_i \to t_\infty$. A further subsequence can be taken to guarantee that either $j_i \to \infty$ or j_i is constant $j_i = j_0$. In the diverging case we use $t_i \le h_{j_i} \to 0$ to see that $t_\infty = 0$, thus our subsequence converges to

$$z_\infty = \chi_\infty(x_\infty) = \zeta_{j_i}(x_\infty,0) \in Z_0 \tag{3.47}$$

which we see as follows:

$$\begin{aligned} d_Z(z_i,z_\infty) &= d_Z(\zeta_{j_i}(x_i,t_i),\zeta_{j_i}(x_\infty,0)) \\ &= d_{Z_{j_i}}((x_i,t_i),(x_\infty,0)) \\ &\le d_\infty(x_i,x_\infty) + |t_i - 0| \to 0. \end{aligned} \tag{3.48}$$

In the constant, $j_i = j_0$ case, our subsequence converges to

$$z_\infty = \zeta_{j_0}(x_\infty,t_\infty) \tag{3.49}$$

which we see as follows:

$$\begin{aligned} d_Z(z_i,z_\infty) &= d_Z(\zeta_{j_0}(x_i,t_i),\zeta_{j_0}(x_\infty,t_\infty)) \\ &= d_{Z_{j_0}}((x_i,t_i),(x_\infty,t_\infty)) \\ &\le d_\infty(x_i,x_\infty) + |t_i - t_\infty| \to 0. \end{aligned} \tag{3.50}$$

Thus Z is compact. □

Lemma 3.3. *Assume the hypotheses of Proposition* 3.3. *For all* $j \in \mathbb{N}$ *there exists*

$$W_j = \bigcup_{k=j}^{\infty} \chi_k(Z_k) \subset Z \tag{3.51}$$

and there is a 1*-Lipschitz map,* $F_j : (W_j,d_Z) \to (X,d_j)$*, where*

$$F_j(\zeta_k(x,t)) = x \qquad \forall k \ge j, \forall x \in X, t \in [0,h_k]. \tag{3.52}$$

Proof. Let $z_1,z_2 \in W_j \subset Z$. Then

$$\exists k_i \ge j,\, x_i \in Xm,\, t_i \in [0,h_{k_i}]m, \text{ s.t. } z_i = \chi_{k_i}(x_i,t_i). \tag{3.53}$$

If $k_1 = k_2$, since $d_\infty \ge d_{k_i} \ge d_j$, by the minimum in the definition of $d_{Z_{k_i}}$ and the monotonicity of the distance functions, we get

$$d_Z(z_1,z_2) \ge d_{k_i}(x_1,x_2) \ge d_j(x_1,x_2). \tag{3.54}$$

If $k_2 \neq k_1$, then observe that for all $x \in X$,

$$d_{Z_{k_i}}(z_i,(x,0)) = d_{Z_{k_i}}((x_i,t_i),(x,0)) \geq d_{k_i}(x_i,x) \geq d_j(x_i,x) \tag{3.55}$$

by the definition of $d_{Z_{k_i}}$ because $t_i \geq 0$ and $h_{k_i} - t_i + h_{k_i} \geq 0$, followed by the monotonicity of the distance functions. Thus

$$d_{Z_{k_1}}(z_1,(x,0)) + d_{Z_{k_2}}((x,0),z_2) \geq d_j(x_1,x) + d_j(x_2,x). \tag{3.56}$$

Taking the infimum over all $x \in X$ on the left and applying the triangle inequality for d_j on the right we have,

$$d_Z(z_1,z_2) \geq d_j(x_1,x_2). \qquad \square$$

4 Intrinsic flat convergence

In this section we review the definition of integral current spaces which include all oriented Riemannian manifolds of finite volume with boundary of finite volume. We then review the Sormani-Wenger Intrinsic Flat (SWIF) distance between these spaces. Finally we state and prove Theorem 4.2 concerning the convergence of monotone increasing sequences of integral current spaces which implies the SWIF convergence claimed in Theorem 1.1.

4.1 Lipschitz functions and tuples on metric spaces

We say that a function, $F:(X,d_X) \to (Y,d_Y)$ is K-Lipschitz if there exists $K \geq 0$ such that

$$d_Y(F(x_1,x_2)) \leq K d_X(x_1,x_2) \qquad \forall x_1,x_2 \in X. \tag{4.1}$$

We define the Lipschitz constant of F to be the smallest such K.

Lemma 4.1. *Suppose that (X,d_a) and (X,d_b) are two metric spaces and*

$$d_a(x_1,x_2) \leq d_b(x_1,x_2) \qquad \forall x_1,x_2 \in X. \tag{4.2}$$

If a function $\pi: X \to Y$ is K-Lipschitz as a map $\pi:(X,d_a) \to (Y,d_Y)$ then it is also K-Lipschitz as a map $\pi:(X,d_b) \to (Y,d_Y)$.

Proof.

$$d_Y(\pi(x_1),\pi(x_2)) \leq K d_a(x_1,x_2) \leq K d_b(x_1,x_2) \qquad \forall x_1,x_2 \in X. \qquad \square$$

By the definition in [2] an m-tuple, $(\pi_0,\pi_1,...,\pi_m)$, of Lipschitz functions with respect to d_a consists of a bounded Lipschitz function, $\pi_0:(X,d_a) \to \mathbb{R}$, and m Lipschitz functions, $\pi_k:(X,d_a) \to \mathbb{R}$, for $k=1,\ldots,m$.

Corollary 4.1. *Suppose that* (X,d_a) *and* (X,d_b) *are two metric spaces and*

$$d_a(x,y) \le d_b(x,y) \qquad \forall x,y \in X. \tag{4.3}$$

Then tuples of Lipschitz functions with respect to d_a *are tuples of Lipschitz functions with respect to* d_b.

Lemma 4.2. *Given the hypotheses of Proposition* 3.3 *and Lemma* 3.3. *If* $\pi:(X,d_j)\to Y$ *is Lipschitz* K_j, *and* $F_j:(W_j,d_Z)\to(X,d_j)$ *of Lemma* 3.3, *then*

$$\tilde{\pi}:(W_j,d_Z)\to Y \quad \text{such that } \tilde{\pi}(z)=\pi\circ F_j(z) \tag{4.4}$$

is also Lipschitz K_j *and satisfies*

$$\tilde{\pi}(\chi_\infty(x))=\pi(x) \qquad \forall x\in X. \tag{4.5}$$

Remark 4.1. Note that if we try to define $\tilde{\pi}$ as a function from all of (Z,d_Z) to $\mathbb{R}$ as in (4.4) then it might not be Lipschitz. The proof depends strongly on the restriction to $W_j\subset Z$ and Lemma 3.3.

Proof. Recall $F_j:(W_j,d_Z)\to(X,d_j)$ are Lipschitz one functions, so

$$d_Y(\tilde{\pi}(z_1),\tilde{\pi}(z_2))\le K_j d_j(F_j(z_1),F_j(z_2))\le K_j d_Z(z_1,z_2) \qquad \forall z_1,z_2\in Z. \qquad \square$$

4.2 Charts into our metric space

We say that a map $\psi:U\subset\mathbb{R}^m\to X$ from a Borel subset U in $\mathbb{R}^m$ into a metric space, (X,d_X), is a Lipschitz chart if there exists $K_\psi>0$ such that

$$d_X(\psi(a_1),\psi(a_2))\le K_\psi|a_1-a_2| \qquad \forall a_1,a_2\subset U. \tag{4.6}$$

These charts will be applied in the next section to define rectifiable and integral currents, and then integral current spaces in the following section.

Lemma 4.3. *Suppose* (X,d_a) *and* (X,d_b) *are two metric spaces and*

$$d_a(x,y)\le d_b(x,y) \qquad \forall x,y\in X. \tag{4.7}$$

If $\psi:U\subset\mathbb{R}^m\to X$ *is a Lipschitz chart with respect to* d_b *on* X, *then it is Lipschitz with respect to* d_a *on* X.

Proof. We have a Lipschitz constant $K_\psi\in(0,\infty)$ such that

$$d_b(\psi(a_1),\psi(a_2))\le K_\psi|a_1-a_2| \qquad \forall a_1,a_2\in U.$$

By (4.7), we have

$$d_a(\psi(a_1),\psi(a_2))\le K_\psi|a_1-a_2| \qquad \forall a_1,a_2\in U. \qquad \square$$

Remark 4.2. Sometimes in the definition of a rectifiable space or current it is written that the charts $\psi_i : U_i \to X$ are bi-Lipschitz. However, once one has a countable collection of Lipschitz charts, one can replace them with a collection of bi-Lipschitz charts by Lemma 4 in Kirchheim's [14] (Lemma 4.1 of [2]). The bi-Lipschitz charts are not necessarily the same collection of charts. If we apply our Lemma 4.3 to a collection of charts $\psi_i : (U_i \subset \mathbb{R}^m, d_{\mathbb{R}^m}) \to (X, d_b)$ that are bi-Lipschitz onto their images, we conclude they are Lipschitz as charts into (X, d_a) but cannot determine if they are bi-Lipschitz. We would need to apply Kirchheim's theorem to produce a new collection of bi-Lipschitz charts into (X, d_b) if we need them.

4.3 Ambrosio-Kirchheim's integral currents on complete metric spaces

We review the work of Ambrosio-Kirchheim in [2]. They defined m dimensional currents on complete metric spaces, (Z, d_Z) as multilinear functionals, T, acting on Lipschitz tuples satisfying various hypotheses [2]. For example, given a compact m-dimensional oriented Riemannian manifold, (M, g), we can define the current,

$$[[M]](\pi_0, \ldots, \pi_m) = \int_M \pi_0 \, d\pi_1 \wedge \cdots \wedge d\pi_m \tag{4.8}$$

which is well defined because Lipschitz maps are differentiable almost everywhere. The integration is actually defined by integrating over atlas of disjoint oriented charts and taking the sum.

An m dimensional *integer rectifiable current* on a complete metric space, (Z, d_Z), is a multilinear functional, T, defined on Lipschitz tuples on (Z, d_Z) that has a parametrization which is a countable collection of Lipschitz charts with non-zero Lipschitz constants

$$\psi_i : (U_i \subset \mathbb{R}^m, d_{\mathbb{R}^m}) \to (Z, d_Z) \tag{4.9}$$

defined on Borel sets, $U_i \subset \mathbb{R}^m$, with integer multiplicities a_i, such that total Hausdorff measures satisfies

$$\sum_{i=1}^{\infty} |a_i| \mathcal{H}^m_{d_Z}(\psi_i(U_i)) < \infty. \tag{4.10}$$

The integer rectifiable current T is defined by the weighted sum:

$$T = \sum_{i=1}^{\infty} a_i \psi_{i\#}[[U_i]], \tag{4.11}$$

where

$$\psi_\#[[U]](\pi_0, \pi_1, \ldots, \pi_m) = \int_U (\pi_0 \circ \psi) \, d(\pi_1 \circ \psi) \wedge \cdots \wedge d(\pi_m \circ \psi). \tag{4.12}$$

This integral is well defined because the charts are Lipschitz and thus differentiable almost everywhere. The weighted sum of the integrals defining $T(\pi_0, \ldots, \pi_m)$ is finite because the weighted sum of the Hausdorff measures in (4.10), $\max\{\pi_0\}$, and the product

$\mathrm{Lip}(\pi_j)$ for $j=1,\ldots,m$ can be used to bound the integrals. Warning: the Lipschitz charts and the Hausdorff measures depend on the distance function on Z. See Lemma 4.4 below.

Note that if (M^m,g) is a compact oriented Riemannian manifold with boundary, with a smooth atlas of oriented charts $\psi_i:(W_i\subset\mathbb{R}^m,d_{\mathbb{R}^m})\to(M^m,d_g)$ then we can choose $U_i\subset W_i$ such that $\psi_i(U_i)$ are disjoint and their union covers M^m. This defines a natural current structure for M with weight 1, which agrees with integration of differential m forms:

$$[[M]](\pi_0,...,\pi_m)=\sum_{i=1}^N \psi_{i\#}[[U_i]]=\int_M \pi_0 d\pi_1\wedge\cdots\wedge d\pi_m. \tag{4.13}$$

By Stoke's Theorem, for any collection of smooth functions, $\pi_i:M\to\mathbb{R}$, for $i=1,...,m-1$, we have

$$\int_{\partial M}\pi_1 d\pi_2\wedge\cdots\wedge d\pi_{m-1}=\int_M 1 d\pi_1\wedge\cdots\wedge d\pi_{m-1}. \tag{4.14}$$

To be consistent with Stoke's Theorem, Ambrosio-Kirchheim define the boundary of a current, T, as follows

$$\partial T(\pi_1,...,\pi_{m-1})=T(1,\pi_1,...,\pi_{m-1}), \tag{4.15}$$

for any Lipschitz tuple $(\pi_1,...,\pi_{m-1})$ on Z.

They define an *integral current* to be an integer rectifiable current whose boundary is also integer rectifiable. Be warned however that the parametrization of the boundary, ∂T, is not necessarily found by taking the boundaries of the atlas of charts parametrizing, T.

The *mass measure*, $||T||_{d_Z}$, of a current T on a complete metric space, (Z,d_Z), is the smallest finite Borel measure such that

$$T(\pi_0,\pi_1,...,\pi_m)\le\prod_{i=1}^m \mathrm{Lip}(\pi_i)\int_Z|\pi_0|\,||T||_{d_Z} \tag{4.16}$$

for all tuples and the *mass* of T is

$$\mathbf{M}_{d_Z}(T)=||T||_{d_Z}(Z). \tag{4.17}$$

The support of a current is

$$\mathrm{spt}_{d_Z}(T)=\{z\in Z: ||T||_{d_Z}(B_{d_Z}(z,r))>0\,\forall r>0\} \tag{4.18}$$

and the set of an m dimensional current is

$$\mathrm{set}_{d_Z}(T)=\left\{z\in Z:\liminf_{r\to 0}||T||_{d_Z}(B_{d_Z}(z,r))/r^m>0\,\forall r>0\right\}. \tag{4.19}$$

Ambrosio-Kirchheim prove that $\mathrm{set}_{d_Z}(T)$ is rectifiable and that the closure of this set is the support of T,

$$\mathrm{spt}(T)=\overline{\mathrm{set}_{d_Z}(T)}. \tag{4.20}$$

Given a Lipschitz map, $F:(X,d_X)\to(Y,d_Y)$, the pushforward of an integral current, T, on (X,d_X), to an integral current, $F_\#(T)$, on (Y,d_Y) is defined by

$$F_\#(T)(\pi_0,...,\pi_m)=T(\pi_0\circ F,...,\pi_m\circ F) \tag{4.21}$$

for any Lipschitz tuple on (Y,d_Y). To see that $F_\#(T)$ is an integral current, Ambrosio-Kirchheim prove that the mass is bounded as follows:

$$\mathbf{M}_{d_Y}(F_\#(T))\le(\mathrm{Lip}(F))^m\mathbf{M}_{d_X}(T). \tag{4.22}$$

For our purposes we need the following lemma.

Lemma 4.4. *Suppose (X,d_a) and (X,d_b) are two metric spaces and*

$$d_a(x,y)\le d_b(x,y) \qquad \forall x,y\in X. \tag{4.23}$$

If S is an integral current on (X,d_b) then it is an integral current on (X,d_a) defined using the same collection of Lipschitz charts and weights with

$$\mathbf{M}_{d_a}(S)\le\mathbf{M}_{d_b}(S) \quad \textit{and} \quad \mathbf{M}_{d_a}(\partial S)\le\mathbf{M}_{d_b}(\partial S). \tag{4.24}$$

The mass measures also satisfy

$$||S||_{d_a}\le||S||_{d_b} \quad \textit{and} \quad ||\partial S||_{d_a}\le||\partial S||_{d_b}. \tag{4.25}$$

Proof. By the definition of integral current structure on (X,d_b), there is a countable collection of d_b-Lipschitz charts, $\psi_i:U_i\subset\mathbb{R}^m\to X$, and integers a_i such that

$$S(\omega)=\sum_{i=1}^{\infty}a_i\psi_{i\#}[[U_i]](\omega) \tag{4.26}$$

for any m tuple, ω, of d_b-Lipschitz functions. By Lemma 4.3 these charts are also d_a-Lipschitz charts, so the weighted sum in (4.26) is well defined acting on any m tuple, ω, of d_a-Lipschitz functions. Note that the same can be done for ∂S. Thus S is an integral current on (X,d_a).

To estimate the inequalities of the mass measures, recall that in [2] the mass measure of a current, S, on a metric space (X,d) is defined as the smallest Borel measure, μ, on X such that

$$|S(\pi_0,\pi_1,...,\pi_m)|\le\prod_{k=1}^{m}\mathrm{Lip}_d(\pi_k)\int_X|\pi_0|d\mu \tag{4.27}$$

for all tuples, $(\pi_0,\pi_1,...,\pi_m)$, of d-Lipschitz functions. By Corollary 4.1, for any tuple, $(\pi_0,\pi_1,...,\pi_m)$, of d_a-Lipschitz functions, we also have (4.27) with $\mu=||S||_{d_b}$. Thus the smallest measure for the tuples of d_a-Lipschitz functions exists and is smaller,

$$||S||_{d_a}\le||S||_{d_b}. \tag{4.28}$$

The same holds for ∂S which implies the total mass estimate in (4.24). □

Note the above lemma implies that distance preserving maps push forward currents conserving their masses, boundaries, and boundary masses.

Remark 4.3. The converse of Lemma 4.4 does not hold in general. If T is an integral current on (X,d_a), it does not necessarily define an integral current on (X,d_b) for $d_b \geq d_a$. A parametrization of T by Lipschitz charts into (X,d_a) might not give Lipschitz charts into (X,d_b) because that would require a converse to Lemma 4.3 (which we do not have). In addition, T does not necessarily act on tuples in (X,d_b) because that would require a converse to Lemma 4.1 (which we do not have). Note that if one could prove that either or both of the above were true for some special T, one would still need to verify that the mass of T and ∂T are finite with respect to d_b to conclude that T is an integral current on (X,d_b).

Soon we will be considering a sequence of metric spaces (X,d_j) with an integral current T, and the limit, (X,d_∞). Since $d_\infty \geq d_j$, we will need to overcome the lack of a converse to Lemma 4.4. We will see later in the proof of Lemma 4.5, that we can sometimes try to rewrite a parametrization of an integral current on (X,d_j) to obtain charts that are also Lipschitz into (X,d_∞). We will see later in the proof of Theorem 4.2, which is needed to complete the proof of Theorem 1.1, that we can sometimes find an integral current, T_∞, on (X,d_∞) and later show $T_\infty = T$ in the following sense:

Definition 4.1. *If we have an integral current T on (X,d_a) and there is also an integral current, S, on (X,d_b) where $d_b \geq d_a$ such that*

$$S(\pi_0,\pi_1,...,\pi_m) = T(\pi_0,\pi_1,...,\pi_m) \tag{4.29}$$

for any tuple of d_a Lipschitz functions on X, then we say that "$S=T$ viewed as an integral current on (X,d_a)". That is, we can use any parametrization of S and ∂S with charts into (X,d_b) as a parametrization for T and ∂T with the same weights and charts into (X,d_a) respectively. †

4.4 Weak convergence of currents

A sequence T_j *converges weakly as currents* in (Z,d_Z) to an integral current, T_∞ iff

$$T_j(\pi_0,\pi_1,...,\pi_m) \to T_\infty(\pi_0,\pi_1,...,\pi_m) \tag{4.30}$$

for all tuples, $(\pi_0,\pi_1,...,\pi_m)$ on (Z,d_Z).

We will apply the following powerful theorem of Ambrosio-Kirchheim:

Theorem 4.1 (Ambrosio-Kirchheim)**.** *If Z is compact and T_j have uniform upper bounds*

$$\mathbf{M}(T_j) \leq V_0 \quad \text{and} \quad \mathbf{M}(\partial T_j) \leq A_0 \tag{4.31}$$

then a subsequence T_{j_k} converges weakly to an integral current, T_∞, and the mass measure is lower semicontinuous:

$$||T_\infty||(U) \leq \liminf_{k\to\infty} ||T_{j_k}||(U) \tag{4.32}$$

for any open set $U \subset Z$.

†Note that Lemma 4.4 applies to S but not T as discussed in Remark 4.3.

4.5 Integral current spaces

In [22], Sormani and Wenger define an m-dimensional *integral current space*, (X,d,T), as a metric space, (X,d), with an integral current, T, defined on the closure of (X,d) such that the $\mathrm{set}(T)=X$ where

$$\mathrm{set}(T)=\left\{x\in\bar{X}:\liminf_{r\to 0}||T||_d(B_d(x,r))/r^m>0\right\}. \tag{4.33}$$

Such metric spaces (X,d) are rectifiable and parametrized by the charts that parametrize their integral current structure, T.

Given a smooth compact oriented m-dimensional Riemannian manifold with boundary, (M,g), it can be viewed as an integral current space, $(M,d_g,[[M]])$. Here the mass measure is the volume measure,

$$\mathbf{M}_{d_g}(M)=\mathrm{vol}_g(M), \tag{4.34}$$

and

$$\mathrm{set}_{d_g}([[M]])=\left\{x\in M:\liminf_{r\to 0}\mathrm{vol}_g(B_{d_g}(x,r))/r^m>0\right\}=M. \tag{4.35}$$

The following lemma depicted in Fig. 4 is useful for constructing examples:

Lemma 4.5. *Suppose the sphere $\mathbb{S}^2$ is endowed with the standard round metric, $g_{\mathbb{S}^2}=dr^2+\sin^2(r)d\theta^2$, defined on a sphere parametrized by $r\in[0,\pi]$ and $\theta\in[0,2\pi]$. We define the possibly singular metric, g_h, induced by a graph, $h\circ r:\mathbb{S}^2\to[0,\infty)$ so that*

$$g_h=dr^2+\sin^2(r)d\theta^2+(h'(r))^2dr^2\geq g_{\mathbb{S}^2}=dr^2+\sin^2(r)d\theta^2 \tag{4.36}$$

where $h(r(x))$ is smooth on $r^{-1}(0,\pi)$, increasing on $[0,\pi/2]$ and decreasing on $[\pi/2,\pi]$ with $h(r)=h(\pi-r)$, but possibly only continuous at the poles p_0,p_π where $r=0,\pi$ respectively. We can define a unique compact metric space, $(\mathbb{S}^2,d_h)$, using g_h-lengths of curves, C, such that

$$C:(0,1)\to\mathbb{S}^2\setminus\{p_0,p_\pi\} \tag{4.37}$$

with bounded radial arclength functions from the poles $p=p_0,p_\pi$,

$$s_p(r(x))=d_h(x,p)=\int_0^{r(x)}\sqrt{1+h'(r)^2}dr \quad \textit{with} \lim_{r\to 0}s_p(r)=0. \tag{4.38}$$

So that $(\mathbb{S}^2,d_h)$ is the metric completion of $(\mathbb{S}^2\setminus\{p_0,p_\pi\},d_{g_h})$.

We can find a parametrization of the standard integral current $[[\mathbb{S}^2]]$ on $(\mathbb{S}^2,d_{\mathbb{S}^2})$ by a countable collection of Lipschitz charts into $(\mathbb{S}^2,d_h)$ which map onto $r^{-1}(0,\pi)$ so that we have an integral current space

$$\left(\mathrm{set}_{d_h}([[\mathbb{S}^2]]),d_h,T_h\right) \quad \textit{where } r^{-1}(0,\pi)\subset\mathrm{set}_{d_h}(T_h)\subset\mathbb{S}^2, \tag{4.39}$$

and where "$T_h=[[\mathbb{S}^2]]$ viewed as an integral current on $(\mathbb{S}^2,d_{\mathbb{S}^2})$" in the sense of Definition 4.1.

So $\mathrm{spt}_{d_h}(T_h)=\mathbb{S}^2$ and $\partial T_h=0$. The poles are included in $\mathrm{set}_{d_h}(T_h)$ iff

$$\lim_{r\to 0}|h'(r)|<\infty. \tag{4.40}$$

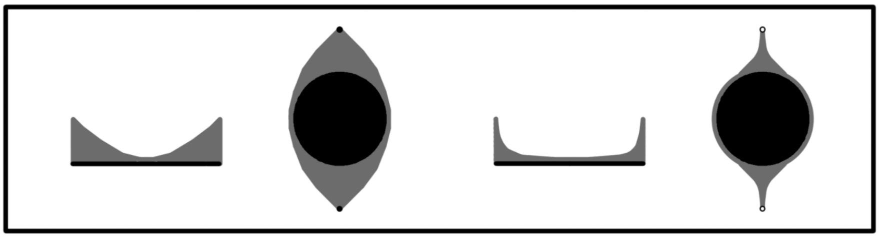

Figure 4: Lemma 4.5 applied with $h=h_{cone}$ of Example 4.1 on the left (which will include the singular poles) and with $h=h_{cusp}$ of Example 4.2 on the right (which will not include the singular poles due to the cusp).

Proof. We easily see that we can define d_h away from the poles using lengths of curves that avoid the poles. Near p_0, we have radial arclengths,

$$s_{p_0}(x)=\int_0^{r(x)}\sqrt{1+h'(r)^2}dr\leq\int_0^{r(x)}1+|h'(r)|dr<\infty \tag{4.41}$$

by the properties of h. So

$$\lim_{r\to 0}s_{p_0}(r)=\lim_{r\to 0}r+|h(r)-h(0)|=0 \tag{4.42}$$

and similarly for p_π. This allows us to define $(\mathbb{S}^2,d_h)$ as in (4.37) so that the identity map to the standard round sphere is a homeomorphism and so $(\mathbb{S}^2,d_{\mathbb{S}^2})$ is compact with bounded diameter.

The usual integral current, $[[\mathbb{S}^2]]$, on the standard round sphere $(\mathbb{S}^2,d_{\mathbb{S}^2})$ can be defined using the standard single (r,θ) chart,

$$\psi:[0,\pi]\times[0,2\pi]\to\mathbb{S}^2, \tag{4.43}$$

with multiplicity one, but this may not be a Lipschitz chart into $(\mathbb{S}^2,d_h)$ due to the singularities at the poles.

Instead we define a new parametrization with a countable collection of multiplicity one charts avoiding the poles,

$$\psi_i:U_i\to\mathbb{S}^2,\quad\text{where } \psi_i(r,\theta)=\psi(r,\theta), \tag{4.44}$$

and where U_i are defined inductively by

$$U_1=[\delta_1,\pi-\delta_1]\times[0,2\pi)\quad\text{ and }\quad U_{i+1}=[\delta_{i+1},\pi-\delta_{i+1}]\times[0,2\pi)\setminus U_i \tag{4.45}$$

where δ_i decrease to 0 so that these are Lipschitz charts in $(\mathbb{S}^2,d_h)$. Then

$$T_h=\sum_{i=1}^{\infty}\psi_{i\#}[[U_i]] \tag{4.46}$$

is a rectifiable current on $(\mathbb{S}^2, d_h)$ because

$$\sum_{i=1}^{\infty} \mathcal{H}^2(\psi_i(U_i)) \leq \mathrm{vol}_{g_h}(\mathbb{S}^2) < \infty. \tag{4.47}$$

We see that "$T_h = [[\mathbb{S}^2]]$ viewed as an integral current on $(\mathbb{S}^2, d_{\mathbb{S}^2})$" in the sense of Definition 4.1 because

$$T_h(\pi_0, \pi_1, \pi_2) = [[\mathbb{S}^2]](\pi_0, \pi_1, \pi_2) \tag{4.48}$$

for any tuple in $(\mathbb{S}^2, d_{\mathbb{S}^2})$, since missing two points does not affect the integration of tuples. So $(\mathrm{set}_{d_h}(T_h), d_h, T_h)$, is an integral current space. Furthermore

$$\partial T_h = \partial[[\mathbb{S}^2]] = 0. \tag{4.49}$$

In order to determine which points lie in $\mathrm{set}_{d_h}(T_h)$ we use (4.19) and the fact that the mass measure is the volume measure as stated above (4.34). It is easy to see that points where g_h is smooth lie in the set, so we conclude (4.39) holds. Taking either pole $p \in \{p_0, p_\pi\}$ and $r = r_p$, and applying L'Hopital's rule twice, we have

$$\begin{aligned} \lim_{R\to 0} \tfrac{1}{R^2} \mathrm{vol}_{g_h}(B_{d_h}(p,R)) &= \lim_{R\to 0} \tfrac{1}{R^2} \int_0^R 2\pi r(s)\, ds = \lim_{s\to 0} \frac{2\pi r(s)}{2R} \\ &= \lim_{s\to 0} \pi r'(s) = \lim_{s\to 0} \pi (1 + (h'(r))^2)^{-1/2}. \end{aligned} \tag{4.50}$$

So $p \in \mathrm{set}_{d_0}(T_h)$ iff this limit is > 0 which happens iff (4.40) holds. □

Example 4.1. If we consider g_h of Lemma 4.5 defined using

$$h(r) = h_{cone}(r) = 1 - |\sin(r)| \tag{4.51}$$

then $(\mathbb{S}^2, d_{cone})$ is a compact metric space with conical singularities, and

$$(\mathrm{set}_{d_{cone}}([[\mathbb{S}^2]], d_h, [[\mathbb{S}^2]]))$$

is an integral current space with

$$\mathrm{set}([[\mathbb{S}^2]]) = \mathbb{S}^2. \tag{4.52}$$

as depicted in Fig. 4 because

$$\lim_{r\to 0} |h'(r)| = \lim_{r\to 0} \cos(r) = 1 < \infty. \tag{4.53}$$

Example 4.2. If we consider $g_{cusp} = g_h$ of Lemma 4.5 defined using

$$h(r) = h_{cusp}(r) = 1 - \sqrt{\sin(r)} = 1 - (\sin^2(r))^{1/4} \tag{4.54}$$

then $(\mathbb{S}^2, d_{cusp})$ is a compact metric space with cusp singularities at the poles, and the triple $(\mathrm{set}_{d_{cusp}}([[\mathbb{S}^2]]), d_h, [[\mathbb{S}^2]])$ is an integral current space with

$$\mathrm{set}_{d_{cusp}}([[\mathbb{S}^2]]) = \mathbb{S}^2 \setminus \{p_0, p_\pi\}. \tag{4.55}$$

as depicted in Fig. 4 because

$$\lim_{r\to 0}|h'(r)| = \lim_{r\to 0}\frac{1}{2}\frac{\cos(r)}{\sqrt{\sin(r)}} = \infty. \tag{4.56}$$

Theorem 1.1 is careful with the description of the SWIF limit because the space, $(M, d_\infty, [[M]])$, might not be an integral current space, as it might fail to have $\mathrm{set}([[M]]) = M$ as in Example 4.3 depicted in Fig. 5.

Example 4.3. Let us consider the sphere, $\mathbb{S}^2$ with a sequence of metric tensors, $g_j = g_{h_j}$ as in Lemma 4.5 defined using

$$h_j(r) = 1 + \tfrac{1}{j} - \left(\tfrac{1}{j^4} + \sin^2(r)\right)^{\frac{1}{4}}. \tag{4.57}$$

Note that $h_j \circ r : \mathbb{S}^2 \to [0,1]$ are smooth functions because $sin^2(r)$ is a smooth nonnegative function on a sphere even at the poles where $r = 0, \pi$ and $u^{1/4}$ is smooth for $u > 0$. In fact

$$|h_j'(r)| = \tfrac{1}{4}\left(\tfrac{1}{j^4} + \sin^2(r)\right)^{-\frac{3}{4}} (2\sin(r)\cos(r)). \tag{4.58}$$

so

$$|\nabla(h_j \circ r)| = |h_j'(r)| \le |h_{j+1}'(r)| = |\nabla(h_{j+1})| \le |h_{cusp}'(r)| \tag{4.59}$$

and $\nabla(h_j \circ r)$ has the same direction as ∇r for all $j \in \mathbb{N}$. Thus we have a monotone increasing sequence of metric tensors as described in Lemma 2.2. The radial arclengths,

$$s_j(x) = \int_0^{r(x)} \sqrt{1 + h_j'(r)^2}\, dr \le \int_0^{r(x)} 1 + |h_{cusp}'(r)|\, dr, \tag{4.60}$$

are uniformly bounded, so we have a uniform upper bound on diameter.

Observe that $h_j \to h_{cusp}$ of Example 4.2 smoothly away from the poles. So $d_j \to d_{cusp}$ pointwise away from the poles. Since the arclength parameters also converge, we see that $d_j \to d_{cusp}$ on $\mathbb{S}^2 \times \mathbb{S}^2$. Since $(\mathbb{S}^2, d_{\mathbb{S}^2})$ is compact we have all the hypotheses of Theorem 1.1. This is an example where the SWIF limit is a proper subset of the GH limit:

$$\mathrm{set}_{d_{cusp}}([[\mathbb{S}^2]]) \ne \mathbb{S}^2. \tag{4.61}$$

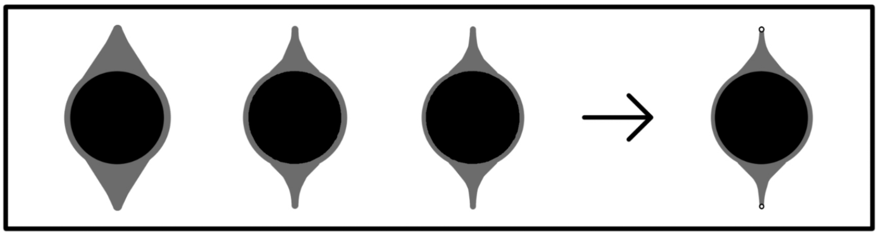

Figure 5: Example 5 has $d_j \le d_{j+1}$ because $|h'_j| \le |h'_{j+1}|$ and converges to Example 4.2 which has the poles with cusp singularities removed.

4.6 Intrinsic flat distance

In [22], Sormani and Wenger define the intrinsic flat distance between two integral current spaces as follows:

Definition 4.2.

$$d_{SWIF}((X_a,d_a,S_a),(X_b,d_b,S_b)) = \inf\{d_F^Z(f_{a\#}(S_a),f_{b\#}(S_b))\} \tag{4.62}$$

where the infimum is taken over all complete metric spaces, (Z,d_Z), and over all distance preserving maps,

$$f_a:(X_a,d_a)\to(Z,d_Z) \quad \text{and} \quad f_b:(X_b,d_b)\to(Z,d_Z). \tag{4.63}$$

Here d_F^Z denotes the Flat distance between integral currents T_a,T_b on (Z,d_Z):

$$d_F^Z(T_a,T_b) = \inf\{\mathbf{M}(A)+\mathbf{M}(B): A+\partial B = T_a - T_b\} \tag{4.64}$$

where the infimum is taken over all integral currents A,B on (Z,d_Z).

We say that a sequence of integral current spaces converges in the intrinsic flat sense,

$$(X_j,d_j,S_j) \xrightarrow{\text{SWIF}} (X_\infty,d_\infty,S_\infty), \tag{4.65}$$

if

$$d_{SWIF}((X_j,d_j,S_j),(X_\infty,d_\infty,S_\infty)) \to 0. \tag{4.66}$$

In [22], Sormani-Wenger prove this implies there is a common complete metric space, (Z,d_Z), and distance preserving maps, $f_j: X_j \to Z$ such that, $T_j = f_{j\#}S_j$ converge weakly to "$T_\infty = f_{\infty\#}S_\infty$ as integral currents on (Z,d_Z)". Thus by Ambrosio-Kirchheim semicontinuity of mass,

$$\liminf_{j\to\infty} \mathbf{M}_{d_j}(S_j) = \liminf_{j\to\infty} \mathbf{M}_{d_Z}(T_j) \ge \mathbf{M}_{d_Z}(T_\infty) = \mathbf{M}_{d_\infty}(S_\infty). \tag{4.67}$$

We say the sequence of integral current spaces converges in the *volume preserving intrinsic flat sense* if we have (4.65) and

$$\mathbf{M}_{d_j}(S_j) \to \mathbf{M}_{d_\infty}(S_\infty). \tag{4.68}$$

This notion was first studied by Portegies in [16]. Additional work was completed by Jauregui-Lee in [12]. The consequences of $\mathcal{VF}$ convergence are most recently reviewed by Jauregui-Perales-Portegies in [13].

4.7 Intrinsic Flat Convergence Theorem

We can now state our SWIF convergence theorem:

Theorem 4.2. *Suppose (X,d_j) is a monotone increasing sequence of metric spaces satisfying all the hypotheses of Theorem 3.1 so that $d_j \to d_\infty$ uniformly and*

$$(X,d_j) \xrightarrow{GH} (X,d_\infty) \text{ which is compact.} \tag{4.69}$$

If there is a current, T, which is an integral current structure for (X,d_j),

$$\mathrm{set}_{d_j}(T) = X \qquad \forall j \in \mathbb{N}, \tag{4.70}$$

with uniform upper bounds on total mass,

$$\mathbf{M}_{d_j}(T) \le V_0 \quad \text{and} \quad \mathbf{M}_{d_j}(\partial T) \le A_0 \qquad \forall j \in \mathbb{N}, \tag{4.71}$$

then

$$(X,d_j,T) \xrightarrow{\mathcal{F}} (M_\infty,d_\infty,T_\infty). \tag{4.72}$$

where "$T_\infty = T$ viewed as an integral current on (X,d_j)" in the sense defined in Definition 4.1 and

$$M_\infty = \mathrm{set}_{d_\infty}(T_\infty) \subset X \text{ with } \overline{M}_\infty = X. \tag{4.73}$$

Furthermore,

$$\mathbf{M}_{d_\infty}(T_\infty) = \lim_{j\to\infty} \mathbf{M}_{d_j}(T). \tag{4.74}$$

so we have volume preserving intrinsic flat ($\mathcal{VF}$) convergence.

Recall that in Example 4.3 we saw that the SWIF limit, M_∞, might be a proper subset of the Gromov-Hausdorff limit, X.

4.8 The proof of Riemannian Theorem 1.1

Before we prove Theorem 4.2, we show in this subsection how it can be applied to prove Theorem 1.1:

Proof. Note that the hypotheses of Theorem 1.1 imply the hypotheses of Theorems 3.1 and 4.2 as follows:

- Monotonicity of metric tensors in (1.1) implies monotonicity of distances as in Definition 2.2 by Lemma 2.1.

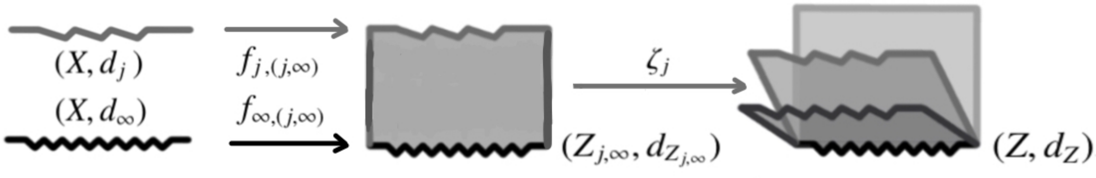

Figure 6: The distance preserving maps from Proposition 3.3.

- Adding the diameter bound in (1.2) implies the pointwise convergence of $d_j \to d_\infty$ in Lemma 2.3 and the diameter hypothesis of Theorem 3.1.
- The assumption that (M,d_∞) is compact in Theorem 1.1 implies the compactness assumption in Theorem 3.1, so that we have uniform convergence as in Proposition 3.1 and a common metric space Z_j for (M,d_j) and (M,d_∞) as in Proposition 3.3 and $(M,d_j) \xrightarrow{\text{GH}} (M,d_\infty)$ as in Theorem 3.1.
- The hypothesis that (M,g_j) are smooth oriented Riemannian manifolds in Theorem 1.1 implies $(M,d_{g_j},[[M]])$ are integral current spaces with $T=[[M]]$ by (4.35).
- The volume bounds in (1.4) imply the mass bounds in (4.71) by (4.34).

Thus we may apply the Theorem 4.2 and (4.34) to conclude the volume preserving intrinsic flat convergence in (1.5) to a limit space satisfying (1.6). $\square$

4.9 Finding the current structure of the SWIF limit for Theorem 4.2

In [22], Sormani-Wenger proved that when one has a sequence of integral current spaces which converge in the Gromov-Hausdorff sense and have uniformly bounded total mass as in (4.71), then SWIF limit of the integral current spaces is a subset of the Gromov-Hausdorff limit. For Theorem 4.2 we wish to show the closure of the SWIF limit is the GH limit, so we need stronger control on the SWIF limit's current structure. To achieve this, we will repeat the steps of the argument in [22] using the special properties of the common metric space, (Z,d_Z), that we constructed in Proposition 3.3. See Fig. 6.

Proposition 4.1. *Under the hypotheses of Theorem* 4.2*, we have the metric space* (Z,d_Z) *as constructed in Proposition* 3.3 *depicted in Fig.* 6 *with distance preserving maps,*

$$\chi_j = \zeta_j \circ f_{j,(j,\infty)} : (X,d_j) \to (Z_{j,\infty},d_{Z_{j,\infty}}) \to (Z,d_Z), \tag{4.75}$$

$$\chi_\infty = \zeta_j \circ f_{\infty,(j,\infty)} : (X,d_\infty) \to (Z_{j,\infty},d_{Z_{j,\infty}}) \to (Z,d_Z), \tag{4.76}$$

such that the pushforwards

$$T_j = \chi_{j\#}(T) = \zeta_{j\#}(f_{j\#}(T)) \tag{4.77}$$

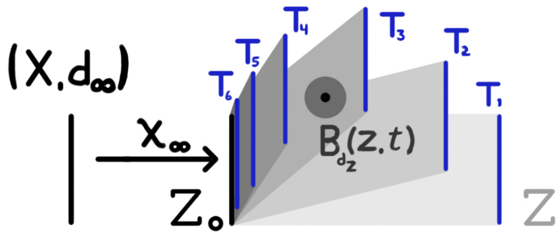

Figure 7: Here we see integral currents T_j in Z and $B_{d_Z}(z,t)$ of the proof that $\mathrm{spt}(\hat{T}_\infty)\subset Z_0=\chi_\infty(X)\subset Z$ as claimed in (4.82).

are integral currents in Z whose supports are

$$\mathrm{spt}(T_j)=\zeta_j(f_j(X))=\zeta_j(X\times\{h_j\})\subset\zeta_j(Z_{j,\infty})\subset Z \tag{4.78}$$

and have

$$\mathbf{M}_{d_Z}(T_j)=\mathbf{M}_{d_j}(T)\le V_0 \quad \textit{and} \quad \mathbf{M}_{d_Z}(\partial T_j)=\mathbf{M}_{d_j}(\partial T)\le A_0. \tag{4.79}$$

Furthermore, there is a subsequence T_{j_k} which converges weakly as currents in (Z,d_Z) to an integral current, $\hat{T}_\infty$, whose support lies on the central spine in Z,

$$\mathrm{spt}(\hat{T}_\infty)=Z_0=\chi_\infty(X)=\zeta_j(X\times\{0\})\subset\zeta_j(Z_{j,\infty})\subset Z. \tag{4.80}$$

Taking $T_\infty=\chi_{\infty\#}^{-1}\hat{T}_\infty$ we have "$T_\infty=T$ viewed as an integral current on (X,d_j)" as in Definition 4.1, and

$$\mathbf{M}_{d_\infty}(T_\infty)=\lim_{k\to\infty}\mathbf{M}_{d_{j_k}}(T). \tag{4.81}$$

In the next section we will apply this proposition to prove SWIF convergence holds without needing a subsequence.

Proof. Since the maps in (4.75) are distance preserving, the pushforward integral currents T_j on (Z,d_Z) defined as in (4.77), satisfy (4.79) and (4.78) following the definitions of pushforward and mass ([22]).

Since (Z,d_Z) is compact by Proposition 3.3, we can apply the Ambrosio-Kirchheim Compactness Theorem, Theorem 4.1, to conclude that there is a subsequence T_{j_k} which converges weakly as currents in (Z,d_Z) to an integral current, $\hat{T}_\infty$.

First we claim

$$\mathrm{spt}(\hat{T}_\infty)\subset Z_0=\chi_\infty(X)=\zeta_j(X\times\{0\}\subset Z_{j,\infty})\subset Z. \tag{4.82}$$

Take any $z\in Z\setminus Z_0$ as in Fig. 7. Then there exists $j_z\in\mathbb{N}$ such that $z=\zeta_{j_z}(f_{j_z}(x,t))$ where $t>0$. Then by the definition of d_Z in Proposition 3.3, the ball:

$$B_{d_Z}(z,t)\subset\zeta_{j_z}(f_{j_z}(X\times(0,h_{j_z}]))) \subset\zeta_{j_z}(Z_{j_z})\setminus Z_0. \tag{4.83}$$

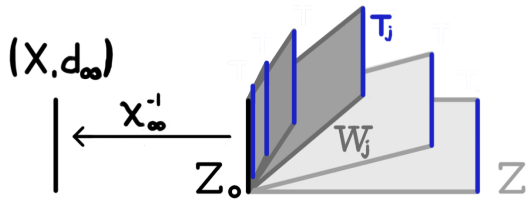

Figure 8: Here we see $W_j \subset Z$ of Lemma 3.3 in dark gray, the rest of Z is lighter, and $Z_0 \subset W_j \subset Z$ is in black with the isometry, χ_∞^{-1}, to (X,d_∞).

In particular

$$B_{d_Z}(z,t) \cap \zeta_j(f_j(X \times \{h_j\})) = \emptyset \tag{4.84}$$

for all $j > j_z$. By the lower semicontinuity of mass under weak convergence as in (4.32) we have

$$||\hat{T}_\infty||(B_{d_Z}(z,t)) \leq \liminf_{j\to\infty} ||T_j||(B_{d_Z}(z,t)) = 0 \tag{4.85}$$

so $z \notin \mathrm{spt}(\hat{T}_\infty)$. Thus $\mathrm{spt}(\hat{T}_\infty) \subset Z_0$. Since $\chi_\infty : (X,d_\infty) \to (Z_0,d_Z)$ is an isometry, we can define:

$$T_\infty = \chi^{-1}_{\infty\#} \hat{T}_\infty \tag{4.86}$$

as an integral current on (X,d_∞).

We claim that for each $j \in \mathbb{N}$, "$T_\infty = T$ viewed as an integral current on (X,d_j)" in the sense of Definition 4.1. Given any tuple, $(\pi_0,\pi_1,...,\pi_m)$ of Lipschitz functions on (X,d_j), we need only show

$$T(\pi_0,\pi_1,...,\pi_m) = T_\infty(\pi_0,\pi_1,...,\pi_m). \tag{4.87}$$

First note that by Lemma 4.1 these π_i are also tuples of Lipschitz functions on (X,d_∞), so $T_\infty(\pi_0,...,\pi_m)$ is well defined.

Recall the set $W_j \subset Z$ from Lemma 3.3, Lemma 4.2, and Remark 4.1 depicted in Fig. 8. By Lemma 4.2, we define a tuple of Lipschitz functions, $(\tilde{\pi}_0,\tilde{\pi}_1,...,\tilde{\pi}_m)$, on the subset, (W_j,d_Z), inside (Z,d_Z), such that

$$\tilde{\pi}_i(\zeta_{j_k}(x,t)) = \pi_i(x) \qquad \forall x \in X, t \in [0,h_{j_k}], j_k \geq j. \tag{4.88}$$

By (4.86) and the fact that

$$\tilde{\pi}_i(\zeta_{j_k}(x,t)) = \pi_i(x) = \tilde{\pi}_i(\chi_\infty(x))) \qquad \forall x \in X, t \in [0,h_{j_k}], j_k \geq j, \tag{4.89}$$

we have by definition of T_∞ and then the definition of pushforward that

$$T_\infty(\pi_0,\pi_1,...,\pi_m) = (\chi^{-1}_{\infty\#}\hat{T}_\infty)(\tilde{\pi}_0 \circ \chi_\infty, \tilde{\pi}_1 \circ \chi_\infty,...,\tilde{\pi}_m \circ \chi_\infty) = \hat{T}_\infty(\tilde{\pi}_0,\tilde{\pi}_1,...,\tilde{\pi}_m). \tag{4.90}$$

Note that for any $j_k \geq j$ we have

$$\mathrm{spt}(T_{j_k}) \subset \zeta_{j_k}(Z_{j_k}) \subset W_j \subset Z. \tag{4.91}$$

So T_{j_k} converge weakly as currents to $\hat{T}_\infty$ in (W_j,d_Z). This means, for any Lipschitz tuple, ω, on (W_j,d_Z), we have

$$T_{j_k}(\omega)\to\hat{T}_\infty(\omega). \tag{4.92}$$

Applying this to $\omega=(\tilde{\pi}_0,\tilde{\pi}_1,...,\tilde{\pi}_m)$, we have

$$T_\infty(\pi_0,\pi_1,...,\pi_m)=\lim_{k\to\infty}T_{j_k}(\tilde{\pi}_0,\tilde{\pi}_1,...,\tilde{\pi}_m). \tag{4.93}$$

On the other hand, by (4.88), we have,

$$\tilde{\pi}_i(\zeta_{j_k}(x,t))=\pi_i(x)=\tilde{\pi}_i(\zeta_{j_k}(f_{j_k}(x)))\qquad\forall x\in X,\, t\in[0,h_{j_k}],\, j_k\geq j. \tag{4.94}$$

Thus $T_{j_k}=\zeta_{j_k\#}f_{j_k\#}T$ and we have

$$T_{j_k}(\tilde{\pi}_0,...,\tilde{\pi}_m)=(\zeta_{j_k\#}f_{j_k\#}T)(\tilde{\pi}_0\circ\zeta_{j_k}\circ f_{j_k},...,\tilde{\pi}_m\circ\zeta_{j_k}\circ f_{j_k})=T(\pi_0,...,\pi_m). \tag{4.95}$$

Substituting this into (4.93), and taking the limit $j_k\to\infty$, we reach our claim in (4.87). We conclude that "$T_\infty=T$ viewed as an integral current on (X,d_j)" in the sense of Definition 4.1.

Now we claim $\mathrm{spt}_{d_Z}(\hat{T}_\infty)=Z_0$. Since we know $\hat{T}_\infty=\chi_{\infty\#}T_\infty$, this is the same as showing $\mathrm{spt}_{d_\infty}(T_\infty)=X$. Given any $x\in X$ and any $r>0$ we have

$$||T_\infty||_{d_\infty}(B_{d_\infty}(x,r))\geq||T_\infty||_{d_j}(B_{d_\infty}(x,r)) \tag{4.96}$$

by Lemma 4.4. By our hypotheses, we have uniform convergence of the distance functions, $|d_j-d_\infty|<\epsilon_j\to 0$, so for all j such that $\epsilon_j<r/2$

$$B_{d_\infty}(x,r)\supset B_{d_j}(x,r/2). \tag{4.97}$$

Combining this with the above, and then the fact that "$T_\infty=T$ as integral currents on (X,d_j)", we have

$$||T_\infty||_{d_\infty}(B_{d_\infty}(x,r))\geq||T_\infty||_{d_j}(B_{d_j}(x,r/2))=||T||_{d_j}(B_{d_j}(x,r/2))>0. \tag{4.98}$$

Thus $\mathrm{spt}_{d_\infty}(T_\infty)=X$ and $\mathrm{spt}_{d_Z}(\hat{T}_\infty)=Z_0$.

Finally, we claim the masses converge as in (4.81). By Ambrosio-Kirchheim's semi-continuity of mass combined with the fact that ζ_∞ is distance preserving we have

$$\mathbf{M}_{d_\infty}(T_\infty)=\mathbf{M}_{d_Z}(\hat{T}_\infty)\leq\liminf_{k\to\infty}\mathbf{M}_{d_Z}(T_{j_k})=\liminf_{k\to\infty}\mathbf{M}_{d_{j_k}}(T). \tag{4.99}$$

Applying Lemma 4.4, and then the fact that "$T_\infty=T$ as integral currents on (X,d_{j_k})", we have

$$\mathbf{M}_{d_\infty}(T_\infty)\geq\mathbf{M}_{d_{j_k}}(T_\infty)=\mathbf{M}_{d_{j_k}}(T). \tag{4.100}$$

Combining (4.99) and (4.100), we have the claimed mass convergence for the subsequence in (4.81). □

4.10 Proving SWIF convergence

In this section we complete the proof of Theorem 4.2. Recall Definition 4.2 of the SWIF distance. Our goal is to provide a constructive proof that

$$d_{SWIF}((X,d_j,T),(M_\infty,d_\infty,T_\infty)) \to 0 \text{ as } j \to \infty \tag{4.101}$$

where $(M_\infty,d_\infty,T_\infty)$ is the integral current space defined using a subsequence in Proposition 4.1.

To complete the proof of Theorem 4.2 we will explicitly estimate the SWIF distances. First we prove a more general proposition that provides a more general estimate on the SWIF distance between a pair of integral current spaces:

Proposition 4.2. *Given two metric spaces, (X,d_a) and (X,d_b), and an $\epsilon > 0$ such that*

$$d_b(x,y) - \epsilon \le d_a(x,y) \le d_b(x,y) \qquad \forall x,y \in X. \tag{4.102}$$

If T_a is an integral current on (X,d_a) and T_b is an integral current on (X,d_b) both of dimension, m, and "$T_a = T_b$ as integral currents on (X,d_a)", then

$$d_{SWIF}((M_a,d_a,T_a),(M_b,d_b,T_b)) \le h 2^{\frac{m+1}{2}} (\mathbf{M}_{d_b}(T_b) + \mathbf{M}_{d_b}(\partial T_b)) \tag{4.103}$$

where $M_a = \mathrm{set}(T_a)$, $M_b = \mathrm{set}(T_b)$, and $h = \epsilon/2$.

Proof. Note that these hypotheses imply the hypotheses of Proposition 3.2; so there exists a metric space

$$(Z_{a,b} = X \times [0,h], d_{Z_{a,b}}), \quad \text{where } h = \epsilon/2, \tag{4.104}$$

with

$$d_{Z_{a,b}}(z_1,z_2) \le d_{taxi}(z_1,z_2) = d_b(x_1,x_2) + |t_1 - t_2|, \tag{4.105}$$

and there are distance preserving maps:

$$f_a : (X,d_a) \to Z_{a,b} \quad \text{ s.t. } f_a(x) = (x,h), \tag{4.106}$$

$$f_b : (X,d_b) \to Z_{a,b} \quad \text{ s.t. } f_b(x) = (x,0). \tag{4.107}$$

We will estimate the SWIF distance by constructing integral currents, A and B, on $Z_{a,b} = X \times [0,h]$ such that

$$\partial B + A = f_{a\#} T_a - f_{b\#} T_b \tag{4.108}$$

and then bounding the masses of A and B.

We have an integral current space, (M,d_b,T_b). So the integral current $T_b \times [0,h]$ is defined on the isometric product space, $(M_b \times [0,h], d_{isom})$ defined in Proposition 3.7 of [17] by Portegies-Sormani. This proposition states that

$$\partial(T_b \times [0,h]) = -\partial T_b \times [[(0,h)]] + T_b \times \partial [[(0,h)]], \tag{4.109}$$

and

$$T_b \times \partial[[(0,h)]] = \psi_{h\#} T_b - \psi_{0\#} T_b \tag{4.110}$$

where $\psi_t(x) = (x,t) \in X \times [0,h]$ and

$$\mathbf{M}_{d_{isom}}(T_b \times [0,h]) = h\mathbf{M}(T_b), \tag{4.111}$$

where d_{isom} is the standard isometric product:

$$d_{isom}((x_1,t_1),(x_2,t_2)) = \sqrt{(d_b(x_1,x_2))^2 + (t_1-t_2)^2}. \tag{4.112}$$

We have $M_b \times [0,h] \subset X \times [0,h] = Z_{a,b}$, however $d_{Z_{a,b}})$ is not the standard isometric product distance. So we rescale the isometric product, compare to the taxi product, and apply (4.105) to see that:

$$\sqrt{2} d_{isom}(z_1,z_2) \geq d_{taxi}(z_1,z_2) \geq d_{Z_{a,b}}(z_1,z_2). \tag{4.113}$$

Note that $T_b \times [0,h]$ is an integral current on the rescaled isometric product space, $(X \times [0,h], \sqrt{2} d_{isom})$, satisfying (4.109)-(4.110) except now the mass rescales:

$$\mathbf{M}_{\sqrt{2}d_{isom}}(T_b \times [0,h]) = 2^{\frac{m+1}{2}} h\mathbf{M}_{d_b}(T_b). \tag{4.114}$$

We can apply the Lipschitz one identity map,

$$\iota : (X \times [0,h], \sqrt{2} d_{isom}) \to (X \times [0,h], \sqrt{2} d_{Z_{a,b}}), \tag{4.115}$$

to define the pushforward integral current on $Z_{a,b}$:

$$B = \iota_\#(T_b \times [[(0,h)]]) \tag{4.116}$$

whose mass satisfies

$$\mathbf{M}_{d_{Z_{a,b}}}(B) \leq 2^{\frac{m+1}{2}} h\mathbf{M}_{d_b}(T_b). \tag{4.117}$$

Similarly, we define

$$A = \iota_\#(\partial T_b \times [[(0,h)]]) \tag{4.118}$$

whose mass satisfies

$$\mathbf{M}_{d_{Z_{a,b}}}(A) \leq 2^{\frac{m}{2}} h\mathbf{M}_{d_b}(\partial T_b). \tag{4.119}$$

By the pushforward of (4.109) we have

$$\partial B = -A + \iota_\#(T_b \times \partial[[(0,h)]]) \tag{4.120}$$

and by the pushforward of (4.110) we have

$$\iota_\#(T_b \times \partial[[(0,h)]]) = \iota_\# \psi_{h\#} T_b - \iota_\# \psi_{0\#} T_b. \tag{4.121}$$

Thus we need only show that

$$f_{a\#} T_a = \iota_\# \psi_{h\#} T_b \tag{4.122}$$

and
$$f_{b\#}T_b = \iota_\# \psi_{0\#} T_b \tag{4.123}$$
as integral currents on $(Z_{a,b}, d_{Z_{a,b}})$. The latter easily follows from,
$$f_b(x) = (x,0) = \iota(x,0) = \iota(\psi_0(x)). \tag{4.124}$$
We also have
$$f_a(x) = (x,h) = \iota(x,h) = \iota(\psi_h(x)). \tag{4.125}$$
which gives
$$f_{a\#}T_a = \iota_\# \psi_{h\#} T_a. \tag{4.126}$$
So we need only prove
$$\iota_\# \psi_{h\#} T_a = \iota_\# \psi_{h\#} T_b \tag{4.127}$$
as integral currents on $(Z_{a,b}, d_{Z_{a,b}})$.

Consider any Lipschitz tuple $(\pi_0, ..., \pi_m)$ on $(Z_{a,b}, d_{Z_{a,b}})$. Recall that
$$\iota_\# \psi_{h\#} T_a(\pi_0, ..., \pi_m) = T_a(\pi_0 \circ \iota \circ \psi_h, ..., \pi_m \circ \iota \circ \psi_h). \tag{4.128}$$
For each Lipschitz function $\pi_i : (Z_{a,b}, d_{Z_{a,b}}) \to \mathbb{R}$ we claim that
$$\pi_i \circ \iota \circ \psi_h : (X, d_a) \to \mathbb{R} \text{ is Lipschitz.} \tag{4.129}$$
This follows from the fact that f_a is distance preserving and so
$$\begin{aligned} |\pi_i(\iota(\psi_h(x_1))) - \pi_i(\iota(\psi_h(x_2)))| &= |\pi_i(x_1,h) - \pi_i(x_2,h)| \\ &\le K d_{Z_{a,b}}((x_1,h),(x_2,h)) \\ &= K d_{Z_{a,b}}(f_a(x_1), f_a(x_2)) \\ &= K d_a(x_1,x_2). \end{aligned} \tag{4.130}$$
So $(\pi_0 \circ \iota \circ \psi_h, ..., \pi_m \circ \iota \circ \psi_h)$ is a Lipschitz tuple on (X, d_a). Combining this with the hypothesis that "$T_a = T_b$ as integral currents on (X, d_a)" we have
$$T_a(\pi_0 \circ \iota \circ \psi_h, ..., \pi_m \circ \iota \circ \psi_h) = T_b(\pi_0 \circ \iota \circ \psi_h, ..., \pi_m \circ \iota \circ \psi_h). \tag{4.131}$$
Thus, by the definition of pushforward, we have (4.127). □

We now apply Propositions 3.1, 4.1, and 4.2 to prove Theorem 4.2:

Proof. First note that the hypotheses of Theorem 4.2 imply that the hypotheses of Proposition 3.1 and Proposition 4.1. By Proposition 3.1, we have uniform convergence of compact metric spaces (X, d_j) to a compact limit (X, d_∞):
$$d_\infty(x,y) - \epsilon_j \le d_j(x,y) \le d_\infty(x,y) \qquad \forall x,y \in X \tag{4.132}$$

where $\epsilon_j \to 0$ as $j \to \infty$. By Proposition 4.1 we have an integral current space,

$$(M_\infty, d_\infty, T_\infty), \quad \text{where } M_\infty = \text{set}(T_\infty) \subset X \tag{4.133}$$

with $Cl(M_\infty) = X$ and "$T_\infty = T$ as integral currents on (X, d_j)" for all $j \in \mathbb{N}$.

Next we apply Proposition 4.2 with

$$(X, d_a, T_a) = (X, d_j, T) \quad \text{and} \quad (X, d_b, T_b) = (X, d_\infty, T_\infty) \tag{4.134}$$

where $\text{set}(T_j) = X$ and $\text{set}(T_\infty) = M_\infty$ and

$$h = h_j = 2\epsilon_j \to 0 \text{ as } j \to \infty. \tag{4.135}$$

Thus by Proposition 4.2, we have

$$d_{SWIF}((X, d_j, T), (M_\infty, d_\infty, T_\infty)) \le 2^{\frac{m+1}{2}} h_j (\mathbf{M}_{d_\infty}(T_\infty) + \mathbf{M}_{d_\infty}(\partial T_\infty)). \tag{4.136}$$

Thus by (4.79) of Proposition 4.1 we have,

$$d_{SWIF}((X, d_j, T), (M_\infty, d_\infty, T_\infty)) \le 2^{\frac{m+1}{2}} h_j (V_0 + A_0) \to 0 \tag{4.137}$$

and we may conclude SWIF convergence of the original sequence.

Finally we apply the fact that $\mathbf{M}_{d_j}(T)$ is monotone increasing by Lemma 4.4 and the fact that a subsequence converges to $\mathbf{M}_{d_\infty}(T_\infty)$ by Proposition 4.1 to conclude that mass converges. Thus we have volume preserving intrinsic flat convergence of the original sequence. $\square$

5 Open problems

Let us consider a monotone increasing sequence of integral current spaces, (X, d_j, T), with $d_j \le d_{j+1}$ with the following three hypotheses

$$\text{diam}_{d_j}(X) \le D_0, \quad \mathbf{M}_{d_j}(T) \le V_0, \quad \mathbf{M}_{d_j}(\partial T) \le A_0. \tag{5.1}$$

By Lemma 2.3, we have a pointwise limit, (X, d_∞). We know that without the compactness assumption in our Theorem 1.1 and Theorem 4.2, that we may not have Gromov-Hausdorff Convergence of the sequence. See Example 2.1.

However, we know that by Wenger's Compactness Theorem of [24] that any sequence of integral current spaces satisfying (5.1) has a subsequence converging to a SWIF limit,

$$(X, d_{j_k}, T) \xrightarrow{\text{SWIF}} (X'_\infty, d'_\infty, T'_\infty), \tag{5.2}$$

where the SWIF limit might be the zero space.

Combining the monotonicity assumptions with (5.1), perhaps one can show one or any of the following:

- the sequence itself SWIF converges without needing a subsequence possibly to a nonzero limit space,
- there is $\mathcal{VF}$ convergence to the limit space,
- there is a distance preserving map from the SWIF limit, (X'_∞, d'_∞), into the pointwise limit, (X, d_∞), with possibly some relationship between the currents, T'_∞ and T.

Note that assuming any of the following additional hypotheses might help:

- The integral current spaces, (X, d_j, T), are Riemannian, $(M, d_{g_j}, [[M]])$, possibly without boundary.
- The integral current spaces, (X, d_j, T), have empty boundary, $\partial T = 0$, so we don't have to use A_j in the proof of SWIF convergence.
- The integral current structure, T, is assumed to be an integral current on the pointwise limit, (X, d_∞).

It would be interesting to find counter examples to any or all of these statements as well.

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