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Volume Gap Theorems for Ricci Nonnegative Metrics and Einstein Metrics

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Abstract. In this note, we prove some gap theorems of asymptotic volume ratio for Ricci nonnegative metrics, and gap theorems of volume for Einstein metrics.

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1 Introduction

Given a complete noncompact Riemannian n-manifold (M^n, g), we can define the volume ratio of a geodesic ball as

$$\mathrm{VR}(x,r) = \frac{\mathrm{Vol}_g B(x,r)}{\omega_n r^n}, \qquad \forall x \in M \text{ and } r \geq 0,$$

and the asymptotic volume ratio as

$$\mathrm{AVR}(M^n, g) := \lim_{r\to\infty} \frac{\mathrm{Vol}_g B(x,r)}{\omega_n r^n}.$$

While $\mathrm{VR}(x,r)$ clearly depends on the choice of base points but $\mathrm{AVR}(M^n, g)$ doesn't depend on the choice of base points at all.

If (M^n, g) has non-negative Ricci curvature, then the famous Bishop-Gromov theorem asserts that the volume ratio of a geodesic ball is a monotone decreasing function in terms

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of its radius $r \geq 0$. In this paper, we always assume $\mathrm{Ric} \geq 0$ unless otherwise explicitly claimed. Thus we always have

$$1 \geq \mathrm{VR}(x,r) \geq \mathrm{AVR}(M^n,g), \qquad \forall x \in M,\ r \in (0,\infty).$$

If $\mathrm{AVR}(M^n,g) = 1$, then both equalities hold, which implies that (M^n,g) is isometric to the Euclidean space $\mathbb{R}^n$ with standard Euclidean metric. More is true that there exists a dimension constant $\epsilon(n)$ small enough such that if $\mathrm{AVR}(M^n,g) \geq 1-\epsilon(n)$, then any complete Ricci flat metric must be flat. This is a fact goes back to Anderson in 1990s. The first named author always believes that there must be a gap theorem for AVR with respect to complete Ricci flat metric.

Conjecture 1.1. *Assume that (M^n,g) is a complete Riemannian n-manifold such that $\mathrm{Ric}(g)=0$. If $\mathrm{AVR}(M^n,g) > \frac{1}{2}$, then (M^n,g) is isometric to $\mathbb{R}^n$.*

In this paper, we will prove some partial results toward these conjectures. We will confirm Conjecture 1.1 in the case when $n=4$.

Theorem 1.1. *Assume that (M^4,g) is a complete Riemannian 4-manifold such that $\mathrm{Ric}(g)=0$. If $\mathrm{AVR}(M^4,g) > \frac{1}{2}$, then (M^4,g) is isometric to $\mathbb{R}^4$.*

In general dimensions, the following theorem holds.

Theorem 1.2. *Assume that (M^n,g) is a complete Riemannian n-manifold such that $\mathrm{Ric}(g) \geq 0$ and $\int_{M^n} |\mathrm{Rm}|^{\frac{n}{2}} \mathrm{dvol}_g \leq C$ for some constant $C \in (0,\infty)$. If $\mathrm{AVR}(M^n,g) > \frac{1}{2}$, then (M^n,g) is isometric to $\mathbb{R}^n$.*

Along the proof, we can also reprove the following gap theorem for positive Einstein metrics. The following result has been proved in [4], see also [6]. Let $\mathbb{S}^n$ be the standard round sphere with constant curvature 1.

Theorem 1.3. *Assume (M^n,g) is an Einstein manifold satisfying $Ric(g)=(n-1)g$. There exists a uniform constant $\varepsilon(n)>0$ such that, if $\mathrm{Vol}(M^n,g) \geq \mathrm{Vol}(\mathbb{S}^n)-\varepsilon(n)$, then (M^n,g) is isometric to $\mathbb{S}^n$.*

2 Preliminaries

2.1 Harmonic radius

We firstly recall the definition of harmonic coordinates.

Definition 2.1. *A coordinate chart $(x^1,\cdots,x^n)$ on a Riemannian manifold (M^n,g) is called harmonic if $\Delta_g x^k = 0$ for all $k=1,\cdots,n$.*